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Flat-Sphere Geometry.

SECOND PAPER.

BY JOHN EIESLAND.

INTRODUCTION.

In a paper published in AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV,* I have extended Lie's Line-Sphere Geometry in 3-space to an odd-dimensional space in which a special self-dual element takes the place of the line in 4-space. Conforming to the notation of this memoir, to which we shall refer in the following pages by the letter A, we denote the odd-dimensional space by S_{n-1} (n-even). The following theorem was proved (A, p. 226):

There exist in $\overline{\overline{S}}_{n-1}\infty^{\frac{n+1\cdot n+2}{2}}$ *contact-transformations which transform the* ∞^n *flat spreads*

$$x_i=ay_i+b_i,\quad z=\Sigma c_iy_i+d,\quad \{q_i=-ap_i+c_i\},\qquad i=1,2,\ldots,\frac{n-2}{2}. \tag{1}$$

into the ∞^n *spheres in* S_{n-1}. *These transformations are obtained by superposing the inverse of the generalized Lie transformation* L *on the contact-transformations that leave the* ∞^n *flats* (1) *invariant. These transformations have the characteristic functions:*

$$\left.\begin{array}{l} 1,\ p_i,\ y_i,\ x_i,\ q_i,\ x_ip_i+y_iq_i,\ x_ky_i-x_iy_k,\ q_kp_i-p_kq_i,\ p_ix_k+q_iy_k,\\ x_ip_k+y_iq_k,\ zy_k-y_k\Sigma q_jy_j-x_k\Sigma p_jy_j,\ zp_k-p_k\Sigma q_jy_j+q_k\Sigma p_jy_j,\ \Sigma q_jy_j,\\ \Sigma x_jq_j,\ \Sigma p_jy_j,\ 2z-\Sigma(x_jp_j+y_jq_j),\ zx_k-y_k\Sigma x_jq_j-x_k\Sigma x_jp_j,\ zq_k+p_k\Sigma x_jq_j\\ -q_k\Sigma x_jq_j,\ z^2-z\Sigma q_jy_j-z\Sigma p_jx_j+\Sigma x_jp_j\cdot\Sigma qyy-\Sigma x_jq_j\cdot\Sigma y_jp_j,\\ \qquad\qquad i,k=1,2,\ldots,\dfrac{n-2}{2},\ i\neq k. \end{array}\right\} \tag{2}$$

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* "On a Flat Spread-Sphere Geometry in Odd-Dimensional Space," pp. 201–228.

The inverse of L is:

$$\left.\begin{aligned} x_i &= -(X_{2i-1}+iX_{2i}) - \frac{P_{2i-1}+iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \quad y_i = -\frac{P_{2i-1}+iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \\ p_i &= \frac{P_{2i-1}-iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \quad q_i = -(X_{2i-1}-iX_{2i}) - \frac{P_{2i-1}-iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \\ z &= \frac{\Sigma(P_{2i-1}+iP_{2i})(X_{2i-1}-iX_{2i})}{1\pm\sqrt{1+\Sigma P_i^2}} - X_{n-1}, \qquad i=1, 2, \ldots, \frac{n-2}{2}. \end{aligned}\right\} \quad (3)$$

The same transformations $L^{-1}G_{\frac{n+1\cdot n+2}{2}}$ *transform the asymptotic curves on an* $n-2$*-spread* M_{n-2} *in* $\overline{\overline{S}}_{n-1}$ *into the lines of curvature on the transform of* M_{n-2} *in* S_{n-1}. *The differential equations of the asymptotic curves on* M_{n-2} *are*

$$dx_i dp_{\frac{n-2}{2}} + dy_i dq_{\frac{n-2}{2}} = 0, \quad dq_k dp_{\frac{n-2}{2}} - dp_k dq_{\frac{n-2}{2}} = 0, \quad \left\{\begin{aligned} i &= 1, 2, \ldots, \frac{n-2}{2} \\ k &= 1, 2, \ldots, \frac{n-2}{2} \end{aligned}\right\}, \quad (4)$$

and those of the lines of curvature on the transform are

$$(dX_i + P_i X_{n-1})\, dP_{n-2} - (dX_{n-2} + P_{n-2} dX_{n-1})\, dP_i = 0, \qquad i=1, 2, \ldots, n-3.$$

I

§ 1. The lines that are here denoted as "asymptotic" are from the standpoint of the flat-sphere geometry in S_{n-1} the analogues of asymptotic lines on a surface in 3-space, inasmuch as through any point on the surface there pass $n-2$ such lines, and to these correspond by the generalized Lie transformation (3) the lines of curvature on the transform.*

The lines of curvature on a spread M_{n-2} in S_{n-1} are not necessarily *coordinate lines* of curvature in the Darboux sense. In fact, as Darboux has shown,† an $n-2$-spread has coordinate lines of curvature if and only if an $n-1$-tuple orthogonal system exists of which the given M_{n-2} is one of the $n-1$ mutually orthogonal surfaces. If a surface has coordinate lines of curvature

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* *I. e., in general;* in the case of special surfaces we may have "bands" of curvature or curvature = spreads, in which case the transform has "bands" of asymptotic curves. Such cases we shall meet with in the present paper.

† Darboux, "Leçons sur les systèmes orthogonaux et les coordinées curvilignes," pp. 133–137 and 176–182. See also a note in *Comptes Rendus*, Vol. CXXVIII, pp. 284–285, entitled "Sur les systèmes orthogonaux," by A. Pellet.

it may be shown that the coordinates X_i and the direction cosines of the normal c_i must satisfy the generalized Olinde Rodrigue formulae

$$\frac{\partial X_i}{\partial \rho_k}+R_k\frac{\partial c_i}{\partial \rho_k}=0, \qquad \begin{array}{l} i=1, 2, \ldots, n-1, \\ k=1, 2, \ldots, n-2, \end{array}$$

that is, on a surface having coordinate lines of curvature the functions $\rho_1, \ldots, \rho_{n-2}$ are such that on a line of curvature ρ_k the function ρ_k only varies while the others remain constant. For 3-space this is always true, while for $n>3$ this condition imposes restrictions on the surface as an individual; while every surface in n-space has lines of curvature, not every surface is a member of an n-tuple orthogonal system of surfaces.

We shall say that a spread M_{n-2} in $\bar{\bar{S}}_{n-1}$ has *coordinate asymptotic lines* whenever its transform by the transformation (3) has coordinate lines of curvature.

Through the researches of Darboux and others we now know a great many orthogonal systems in n-space, and therefore also an indefinite number of surfaces having coordinate lines of curvature. If in an odd space we transform these by (3) we obtain surfaces on which the corresponding lines are coordinate asymptotic lines, but since in general, to real surface-elements in S_{n-1} correspond imaginary elements in $\bar{\bar{S}}_{n-1}$, this method is not always convenient and sometimes even impractical, since the eliminations to be performed are rather complicated if the surface in S_{n-1} is to be obtained in parametric or Cartesian form. The lines of curvature being real on a real surface, the corresponding asymptotic lines will in general be imaginary if the transform is a real surface.

§ 2. We shall represent the tangent flat F_{n-2} to an $n-2$-spread in S_{n-1} by the following equation:

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots+i(\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}})X_{n-2} \\ +(1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0. \qquad (5)$$

the envelope of which is the spread M_{n-2} (A, p. 208):

$$\left.\begin{array}{l} X_1=\dfrac{(\alpha_1+\beta_2)}{2}X_{n-1}-\dfrac{1}{2}(F'_{\alpha_1}+F'_{\beta_1}), \\ X_2=i\dfrac{(\alpha_1-\beta_1)}{2}X_{n-1}+\dfrac{i}{2}(F'_{\alpha_1}-F'_{\beta_1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-1}=\dfrac{\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}-F}{1+\Sigma\alpha_i\beta_i}. \end{array}\right\} \qquad (6)$$

The equations (5) may therefore be considered as the tangential equation of the spread. The equations of the lines of curvature are (A, p. 204):

$$\frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\beta_i}}{dF'_{\beta_{\frac{n-2}{2}}}},\quad \frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\alpha_k}}{dF'_{\beta_{\frac{n-2}{2}}}},\qquad \begin{matrix} i=1,2,\ldots,\frac{n-4}{2}, \\ k=1,2,\ldots,\frac{n-2}{2}, \end{matrix} \tag{7}$$

which may be put in the form, writing $\bar{p}_i$ for F'_{α_i} and $\bar{q}_i$ for F'_{β_i},

$$d\alpha_i d\bar{q}_{\frac{n-2}{2}}-d\alpha_{\frac{n-2}{2}}d\bar{q}_i=0,\quad d\beta_k d\bar{q}_{\frac{n-2}{2}}-d\alpha_{\frac{n-2}{2}}d\bar{p}_k=0. \tag{8}$$

If $\rho_1,\ldots,\rho_{n-2}$ are coordinate lines of curvature we must have

$$\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\bar{q}_{\frac{n-2}{2}}}{\partial\rho_k}-\frac{\partial\alpha_{\frac{n-2}{2}}}{\partial\rho_k}\frac{\partial\bar{q}_i}{\partial\rho_k}=0,\qquad \frac{\partial\beta_i}{\partial\rho_k}\frac{\partial\bar{q}_{\frac{n-2}{2}}}{\partial\rho_k}-\frac{\partial\alpha_{\frac{n-2}{2}}}{\partial\rho_k}\frac{\partial\bar{p}_i}{\partial\rho_k}=0, \tag{9}$$

which may be written

$$\frac{\partial\bar{q}_i}{\partial\rho_k}=\lambda_k\frac{\partial\alpha_i}{\partial\rho_k},\quad \frac{\partial\bar{p}_i}{\partial\rho_k}=\lambda_k\frac{\partial\beta_i}{\partial\rho_k},\qquad \left(\begin{matrix} i=1,2,\ldots,\frac{n-2}{2} \\ k=1,2,\ldots,\frac{n-2}{2} \end{matrix}\right), \tag{10}$$

to which must also be added the $n-2$ conditions

$$\frac{\partial F}{\partial\rho_k}=\sum\left[\bar{p}_i\frac{\partial\alpha_i}{\partial\rho_k}+\bar{q}_i\frac{\partial\beta_i}{\partial\rho_k}\right]. \tag{11}$$

Expressing the conditions for the integrability of (10) we obtain the following system of $\frac{(n-2)(n-3)}{2}$ partial differential equations which must be satisfied by the α's and β's:

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}=0. \tag{12}$$

In the same way, expressing the condition for the integrability of (11) we are led to the system of $\frac{n-2\cdot n-3}{2}$ equations:

$$\sum\left[\frac{\partial\bar{p}_i}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}+\frac{\partial\bar{q}_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}-\frac{\partial p_i}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}-\frac{\partial\bar{q}_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0,$$

which by means of (10) may be reduced to the following:

$$(\lambda_k-\lambda_{k'})\sum\left[\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0. \tag{13}$$

Assuming that $\lambda_k \neq \lambda_{k'}$ for all values of k and k' * these conditions are

$$\sum\left[\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0. \tag{14}$$

If, therefore, the lines $\rho_1, \rho_2, \ldots, \rho_{n-2}$ are coordinate lines of curvature on M_{n-2}, the functions α_i and β_i must satisfy the conditions (12) and (13′). These conditions being satisfied, we shall show that F is also a solution of (12). We have from (11),

$$\begin{aligned}\frac{\partial^2 F}{\partial\rho_{k'}\partial\rho_k}&=\sum\frac{\partial p_i}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}+\sum\frac{\partial\bar{q}_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\sum\bar{p}_i\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\sum\bar{q}_i\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}}\\&=\lambda_{k'}\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right)+\sum\bar{p}_i\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\sum\bar{q}_i\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}},\end{aligned}$$

hence

$$\begin{aligned}(\lambda_k-\lambda_{k'})\frac{\partial^2 F}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_k}\frac{\partial F}{\partial\rho_{k'}}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial F}{\partial\rho_{k'}}&=\lambda_{k'}(\lambda_k-\lambda_{k'})\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}\right)\\&+\sum\bar{p}_i\left[(\lambda_k-\lambda_{k'})\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}\right]\\&+\sum\bar{q}_i\left[(\lambda_k-\lambda_{k'})\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}\right].\end{aligned}$$

Now, if the condition of integrability (12) and (14) are satisfied, the right side of this equation vanishes identically; hence F is a solution of (12) *q. e. d.* It may be worth while to note that the system (12) is also satisfied by the function $\Sigma\alpha_i\beta_i$, α_i and β_i being a set of particular solutions. We have then the following:

THEOREM I. *If on a surface* (6) *the lines of curvature are coordinate lines, the curvilinear coordinates* α_i, β_i *must satisfy the system of differential equations,*

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}=0, \tag{12}$$

and also the $\frac{n-2\cdot n-3}{2}$ *relations*

$$\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}\right)=0. \tag{14}$$

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*This amounts to assuming that the surface (6) is an $n-2$-spread. If $\lambda_k=\lambda_{k'}$ for *all* values of k and k', the surface is one-dimensional or a curve.

We shall transform the coordinates α_i and β_i putting

$$\left.\begin{aligned} &\alpha_1+\beta_1=2y_1, \quad \alpha_2+\beta_2=2y_3, \ldots, \alpha_{\frac{n-2}{2}}+\beta_{\frac{n-2}{2}}=2y_{n-3}, \\ &i(\alpha_1-\beta_1)=2y_2, \quad i(\alpha_2-\beta_2)=2y_4, \ldots, i(\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}})=2y_{n-2}. \end{aligned}\right\} \quad (16)$$

The equations (10) now become

$$\frac{1}{2}\frac{\partial}{\partial\rho_k}(\bar{p}_i+\bar{q}_i)=\lambda_k\frac{\partial y_{2i-1}}{\partial\rho_k}, \quad \frac{i}{2}\frac{\partial}{\partial\rho_k}(\bar{q}_i-\bar{p}_i)=\lambda_k\frac{\partial y_{2i}}{\partial\rho_k}, \quad i=1, 2, \ldots, \frac{n-2}{2}, \quad (17)$$

and the new coordinates y_i are still solutions of (12). The conditions (14) are

$$\sum\frac{\partial y_i}{\partial\rho_k}\frac{\partial y_i}{\partial\rho_{k'}}=0, \qquad k, k'=1, 2, \ldots, n-2, \ k\neq k'. \quad (18)$$

We write (12) in the form

$$\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}-\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}\frac{\partial\theta_k}{\partial\rho_{k'}}=0, \quad (19)$$

where the H's are defined by the equations

$$\left.\begin{aligned} \frac{\partial\lambda_k}{\partial\rho_{k'}}&=-(\lambda_k-\lambda_{k'})\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}}, \\ \frac{\partial\lambda_{k'}}{\partial\rho_k}&=\quad(\lambda_k-\lambda_{k'})\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}. \end{aligned}\right\} \quad (20)$$

Observing that (18) expresses the $\frac{n-2\,.\,n-3}{2}$ conditions that the y's, considered as functions of the ρ's, shall form a completely orthogonal system in S_{n-2}, we put

$$dy_1^2+dy_2^2+\ldots+dy_{n-2}^2=H_1^2d\rho_1^2+H_2^2d\rho_2^2+\ldots+H_{n-2}^2d\rho_{n-2}^2 \quad (21)$$

where $H_1, \ldots, H_{n-2}$ satisfy the $\frac{n-2\cdot n-3\cdot n-4}{2}$ conditions (Darboux, *loc. cit.*, p. 165).

$$f_{kk'}(H_{k''kk'})=\frac{\partial^2H_{k''}}{\partial\rho_k\partial\rho_{k'}}-\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}}\frac{\partial H_{k''}}{\partial\rho_k}-\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}\frac{\partial H_{k''}}{\partial\rho_{k'}}=0. \quad (22)$$

$n-2$ such functions being found, the functions λ_k may be found by integrating the system (20); $\bar{p}_i$, $\bar{q}_i$ and F are then determined by quadratures from (17) and (11). The systems (20), which determine the λ's, are equivalent to a single system

$$\frac{\partial\lambda_k}{\partial\rho_{k'}}=-(\lambda_k-\lambda_{k'})\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}},$$

which admits of one and only one set of solutions $\lambda_1\ldots\lambda_{n-2}$, such that

$\lambda_k = f_k(\rho_k)$ for $\rho_k = \rho_k^0$ where ρ_k^0 is an initial value of ρ_k (Darboux, *loc. cit.*, p. 350). We have then

THEOREM II. *The determination of a surface in S_{n-1} on which the lines of curvature are coordinate lines depends on the determination of a completely orthogonal system in a space S_{n-2} of one dimension less. Such a system being given the corresponding surface is found by integrating the system*

$$\frac{\partial \lambda_k}{\partial \rho_{k'}} = -(\lambda_k - \lambda_{k'}) \frac{1}{H_k} \frac{\partial H_k}{\partial \rho_{k'}}, \tag{20'}$$

and by quadratures.

The first part of this theorem has been proved by Darboux * by a different method, starting with the generalized Olinde Rodrigue's formulae for n-space. The method used here will be of advantage in the transformation to the space $\bar{\bar{S}}_{n-2}$.

Instead of determining a system of λ's from (20′), we may take any solution whatever of (19). Let Θ be such a solution; then putting $F = \Theta$ we have at once a surface on which the ρ's are coordinate lines of curvature. In case a general solution of (19) can be found, and such solution always exists since the conditions (22) are satisfied, we obtain all the surfaces having the same spherical representation of the lines of curvature.

APPLICATIONS.

§ 3. Let $H_i = 1$. Then $\frac{\partial \lambda_k}{\partial \rho_k} = 0$, or, $\lambda_k = \phi_k''(\rho_k)$. Equations (19) are now

$$\frac{\partial^2 \theta}{\partial \rho_k \partial \rho_{k'}} = 0, \tag{23}$$

the general solution of which is

$$\theta = \phi_1 + \phi_2 + \ldots + \phi_{n-2},$$

ϕ_i being a function of ρ_i alone. Taking as particular solutions of (23) the $n-2$ independent functions $y_i = \rho_i$ we have by integrating the system (17),

$$\bar{q}_i + \bar{p}_i = 2\phi_{2i-1}', \quad i(\bar{q}_i - \bar{p}_i) = 2\phi_{2i}', \qquad i = 1, 2, \ldots, \frac{n-2}{2},$$

and integrating (11) we find

$$F = 2\sum_1^{n-2} \phi_i + 2c,$$

* Darboux, *loc. cit.*, pp. 178–182.

which shows that F is a general solution of (23). The tangential equation of the surface is, therefore,

$$2\rho_1 X_1 + 2\rho_2 X_2 + \ldots + (1-\Sigma\rho_i^2) X_{n-1} + 2\Sigma\phi_i + 2c = 0. \tag{24}$$

Substituting the values of the α's in terms of the ρ's in (6) we have the surface

$$\left.\begin{aligned} &X_{2i-1} = -\phi'_{2i-1} + \rho_{2i-1} X_{n-1}, \quad X_{2i} = -\phi'_{2i} + \rho_{2i} X_{n-1}, \\ &X_{n-1} = \frac{2\Sigma\phi'_{2i-1}\rho_{2i-1} + 2\Sigma\phi'_{2i}\rho_{2i} - 2\Sigma\phi_i - 2c}{1+\Sigma\rho_i^2}, \\ &\qquad\qquad i = 1, 2, \ldots, \frac{n-2}{2}. \end{aligned}\right\} \tag{25}$$

These equations show that the surfaces have their lines of curvature plane in all the $n-2$ systems. The spherical representation of these lines are circles passing through the point $x_1 = x_2 = \ldots = x_{n-2} = 0$, $x_{n-1} = 1$, as readily appears from (24). The focal surface has $n-2$ sheets, viz.:

$$\bar{X}_i = \rho_i \phi''_k - \phi'_i, \quad \bar{X}_{n-1} = \frac{1-\Sigma\rho_i^2}{2}\phi''_k + \Sigma\rho_i\phi'_i - \Sigma\phi_i - c, \qquad i, k = 1, 2, \ldots, n-2,$$

where the subscript k indicates the individual sheet.*

As a particular case let us put $\phi_{2i-1} = \frac{k+2}{4}\rho_{2i-1}^2$, $\phi_{2i} = \frac{k-2}{4}\rho_{2i}^2$, $c = \frac{k}{4}$. Substituting in (24) and differentiating with respect to $\rho_1, \rho_2 \ldots \rho_{n-2}$ in succession, we have

$$2X_{2i-1} - 2\rho_{2i-1}X_{n-1} = -(k+2)\rho_{2i-1}, \quad 2X_{2i} - 2\rho_{2i}X_{n-1} = -(k-2)\rho_{2i}. \tag{27}$$

Eliminating the ρ's from (24) and (27) we have the cubic surface

$$2X_{n-1}\sum_1^{n-1} X_i^2 - (k-2)\sum_1^{\frac{n-2}{2}} X_{2i-1}^2 - (k+2)\sum_1^{\frac{n-2}{2}} X_{2i}^2 - kX_{n-1}^2 + \frac{k^2-4}{2}X_{n-1} + \frac{k(k^2-4)}{4} = 0, \tag{27'}$$

a special kind of cyclide which we shall study later on.

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*The equation of the focal surface may be obtained from equations (19) A, p. 207, by introducing the variables y, instead of the α's and β's. We obtain the following:

$$\bar{X}_i = \frac{y_i\sigma_k}{2} - \frac{p_i}{2}, \quad \bar{X}_{n-1} = \frac{1-\Sigma y_i^2}{4}\sigma_k + \frac{1}{2}\Sigma p_i y_i - \frac{F}{2}, \qquad i, k = 1, 2, \ldots, n-2,$$

where σ_k is a root of the equation

$$\begin{vmatrix} p_{11}-\sigma & p_{12} \ldots\ldots p_{1n-2} \\ p_{21} & p_{22}-\sigma \ldots\ldots p_{2n-2} \\ \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \\ p_{n-21} & p_{n-22} \ldots\ldots p_{n-2n-2}-\sigma \end{vmatrix} = 0. \tag{26}$$

Let H be of the form $\frac{1}{h}$, h being a function of the ρ's. Darboux has shown* that h must be of the form

$$a(\rho_1^2+\rho_2^2+\ldots\ldots+\rho_{n-2}^2)+2a_1\rho_1+2a_2\rho_2+\ldots\ldots+a_{n-2}\rho_{n-2}+b,$$

where the coefficients satisfy the relation $\Sigma a_i^2=ab$. The coordinates y_i expressed in terms of the ρ's are:

$$y_1=\frac{\rho_1+\frac{a_1}{a}}{h},\ldots\ldots,y_{n-2}=\frac{\rho_{n-2}+\frac{a_{n-2}}{a}}{h}. \tag{28}$$

The differential equations for determining the λ's are

$$\frac{\partial\lambda_k}{\partial\rho_{k'}}=\frac{2(\lambda_k-\lambda_{k'})(a\rho_{k'}+a_{k'})}{h},$$

the general integral of which is

$$\lambda_k=h\phi_k''-2\sum_1^{n-2}\phi_i'(a\rho_i+a_i)+2a\sum_1^{n-2}\phi_i;$$

integrating the systems (17) and (19) we find that F is of the form $F=\frac{2\Sigma\phi_i}{h}$, a result which might also have been obtained by integrating the system of differential equations

$$\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{2(a\rho_{k'}+a_{k'})}{h}\frac{\partial\theta}{\partial\rho_k}+\frac{2(a\rho_k+a_k)}{h}\frac{\partial\theta}{\partial\rho_{k'}}=0,$$

of which the y's are particular solutions. The tangential equation of the surface is

$$2\left(\rho_1+\frac{a_1}{a}\right)X_1+2\left(\rho_2+\frac{a_2}{a}\right)X_2+\ldots\ldots+\left(h-\frac{1}{a}\right)X_{n-1}+2\Sigma\phi_i=0, \tag{24'}$$

from which it appears that no new surfaces are obtained, since (24) may be transformed into (24′) by taking as new ρ's linear functions of the old such that $\rho_i=-(a\rho_i'+a_i)$. It may also be observed that the equations (28) define an inversion in the space S_{n-2}.

§ 4. *Surfaces whose Lines of Curvature are Plane in all Systems.*

The surfaces thus far obtained are not the only ones having their lines of curvature plane in all the $n-2$ systems. The spherical representation of the lines of curvature of such surfaces must consist of an $n-2$-fold orthogonal system of circles on the sphere. The orthogonal system corresponding to the lines of curvature on the surface (23) is a special one, the circles lying in $n-2$

* Darboux, *loc. cit.*, pp. 166–168.

systems of planes which meet at $x_1=x_2=\ldots\ldots=x_{n-2}=0$, $x_{n-1}=1$ and all the circles of any one system being tangent to each other at this point. The $n-2$ lines of intersection of the planes of the $n-2$ systems are parallel to the coordinate axes $x_1, x_2, \ldots\ldots, x_{n-2}$ and perpendicular to the x_{n-1} axis. We may obtain a general system in the following manner: We pass an $n-r$-fold system of $(r-1)$-flats through a point on the x_{n-1}-axis at a distance from the origin $=a$, all the flats of the system having as common axis the flat $x_1=x_2=\ldots\ldots=x_{n-r}=0$, $x_{n-1}=a$. We also pass an $r-2$-fold system of $(n-r+1)$-flats through a point on the x_{n-1}-axis at a distance $\frac{1}{a}$ from the origin, the common axis being the flat $x_{n-r+1}=x_{n-r+2}=\ldots\ldots=x_{n-2}=0$, $x_{n-1}=\frac{1}{a}$. *We shall prove that these two systems of flats will determine on the sphere an $n-2$-fold orthogonal system of circles.* This proposition is an extension to n-space of an analogous one for 3-space. If $a=1$ we get the spherical representation of lines of curvature of the surfaces (24), and for $a=0$ the system is analogous to that of meridians and circles of latitude on a sphere in 3-space.

We express the coordinates x_i of the sphere in terms of the curvilinear coordinates θ_i as follows:

$$\left.\begin{aligned}
x_1&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\ldots\ldots\sin\theta_2\sin\theta_1}{1+a\sin\theta_{n-2}\cos_{n-r}},\\
x_2&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\ldots\ldots\sin\theta_2\cos\theta_1}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-r}&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
x_{n-r+1}&=\frac{\sqrt{1-a^2}\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\ldots\cos\theta_{n-3}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-2}&=\frac{\sqrt{1-a^2}\cos\theta_{n-2}\sin\theta_{n-r+1}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\quad x_{n-1}=\frac{a+\cos\theta_{n-r}\sin\theta_{n-2}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},
\end{aligned}\right\}\quad(29)$$

from which it is easily verified that $\Sigma x_i^2=1$. In order to prove that the system is orthogonal we need only calculate the linear element $d\sigma$. A rather long, but not inherently difficult calculation will show that this element is

$$d\sigma^2=E_1d\theta_1^2+E_2d\theta_2^2+\ldots\ldots+E_{n-r}d\theta_{n-r}^2+E_{n-r+1}d\theta_{n-r+1}^2$$
$$+\ldots\ldots+E_{n-3}d\theta_{n-3}^2+E_{n-2}d\theta_{n-2}^2,\quad(30)$$

where the E's have the following values:

$$\left.\begin{aligned}
E_1 &= \frac{(1-a^2)\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_2}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
E_2 &= \frac{(1-a^2)\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_3}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
E_{n-r} &= \frac{(1-a^2)\sin^2\theta_{n-2}}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2}, \quad E_{n-r+1} = \frac{(1-a^2)\cos^2\theta_{n-2}}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
E_{n-r+2} &= \frac{(1-a^2)\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}}{(1+a\sin\theta_{n-2}\cos_{n-r})^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
E_{n-3} &= \frac{(1-a^2)\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}\ldots\cos^2\theta_{n-4}}{(1+a\sin\theta_{n-2}\cos_{n-r})^2},\\
E_{n-2} &= \frac{(1-a^2)}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2}.
\end{aligned}\right\} \quad (31)$$

That the curves θ_i are circles appears from the following systems of equations deduced from (29):

$$\left.\begin{aligned}
x_1 &= \frac{1}{\sqrt{1-a^2}}\,\frac{\sin\theta_{n-r}\ldots\sin\theta_2\sin\theta_1}{\cos\theta_{n-r}}(x_{n-1}-a),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-r} &= \frac{1}{\sqrt{1-a^2}}\,\frac{\sin\theta_{n-r}\cos\theta_{n-r-1}}{\cos\theta_{n-r}}(x_{n-1}-a),\\
x_{n-r+1} &= -\frac{a\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}}{\sqrt{1-a^2}}\left(x_{n-1}-\frac{1}{a}\right),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-2} &= -\frac{a\cos\theta_{n-2}\sin\theta_{n-r+1}}{\sqrt{1-a^2}}\left(x_{n-1}-\frac{1}{a}\right),
\end{aligned}\right\} \quad (32)$$

and that the geometrical construction given at the beginning of this section is correct appears at once from these equations.

Since the coordinates y_i in terms of the x's are given by the formulae:

$$y_1 = \frac{x_1}{1+x_{n-1}}, \quad y_{n-2} = \frac{x_{n-2}}{1+x_{n-1}}, \quad x_1 = \frac{2y_1}{1+\Sigma y_i^2}, \ldots, x_{n-1} = \frac{1-\Sigma y_i^2}{1+\Sigma y_i^2}, \quad (33)$$

we have

$$\left.\begin{aligned}
y_1 &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\ldots\sin\theta_2\sin\theta_1}{1+\cos_{n-r}\sin\theta_{n-2}},\\
y_2 &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\ldots\cos\theta_1}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
y_{n-r} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}}{1+\cos_{n-r}\sin\theta_{n-2}},\\
y_{n-r+1} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
y_{n-2} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\cos\theta_{n-2}\sin\theta_{n-r+1}}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
1-\Sigma y^2 &= \frac{2(a+\cos\theta_{n-r}\sin\theta_{n-2})}{(1+a)(1+\cos\theta_{n-r}\sin\theta_{n-2})},\\
\frac{2}{1+\Sigma y^2} &= \frac{(a+1)(1+\cos\theta_{n-r}\sin\theta_{n-2})}{1+a\cos\theta_{n-r}\sin\theta_{n-2}}.
\end{aligned}\right\}\quad(34)$$

The coordinates y_i satisfy the following system of $\frac{n-2\cdot n-3}{2}$ differential equations:

$$\left.\begin{aligned}
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}-\cot\theta_{k'}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k,k'=1,2,\ldots,n-r-1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}-\tan\theta_{k'}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k,k'=n-r+1,\ldots,n-r),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-r}}-\frac{\cot\theta_{n-r}+\csc\theta_{n-r}\sin\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k=1,2,\ldots,n-r-1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}=0,\quad\left\{\begin{matrix}k=1,2,\ldots,n-r-1\\ k'=n-r+1,\ldots,n-3\end{matrix}\right\},\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-r}}+\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k=n-r+1,\ldots,n-3),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-2}}-\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k=1,2,\ldots,n-r+1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-2}}+\frac{\tan\theta_{n-2}+\cos\theta_{n-r}\sec\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta}=0,\quad(k=n-r+1,\ldots,n-3),\\
&\frac{\partial^2\phi}{\partial\theta_{n-r}\partial\theta_{n-2}}-\frac{\sin\theta_{n-2}\sin\theta_{n-r}}{1+\cos_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_{n-2}}-\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_{n-r}}=0.\quad k\neq k'.
\end{aligned}\right\}\quad(35)$$

The general solution of this system is

$$F=\frac{\Sigma\Phi_i\sqrt{E_i}(1+\Sigma y_i^2)}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\qquad(36)$$

where Φ_i is a function of θ_i alone and the E's have the values given in (31). The surface is therefore the envelope of the flats

$$2y_1X_1+2y_2X_2+\ldots\ldots+2y_{n-r}X_{n-r}+2y_{n-r+1}X_{n-r+1}+\ldots\ldots \\ +(1-\Sigma y_i^2)X_{n-1}=(1+\Sigma y_i^2)\Sigma\Phi_i\sqrt{E_i}. \quad (37)$$

For any constant value of r all the surfaces obtained by putting for Φ_i arbitrary functions of θ_i have the same spherical representation of their lines of curvature. Since r can have any value from 3 to $n-1$ *there will be* $n-3$ *different types.* To these types must also be added the type (24) corresponding to $a=1$ obtained in § 3, viz.:

$$2\rho_1X_1+2\rho_2X_2+\ldots\ldots+2\rho_{n-r}X_{n-r}+\ldots\ldots+(1-\Sigma\rho_i^2)X_{n-1}+2\Sigma\phi_i+2c=0. \quad (24)$$

The surfaces (37) may be considered as the envelope of the radical flats of the two spheres:

$$\left.\begin{aligned}
&\sum_1^{n-1} X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\ldots\cos\theta_{n-3}X_{n-r+1}+\ldots\ldots \\
&\qquad +2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{n-2}+\frac{2a\csc\theta_{n-2}}{\sqrt{1-a^2}}X_{n-1}=2\Phi_{n-2}\csc\theta_{n-2} \\
&\qquad +2\Phi_{n-3}\cot\theta_{n-2}\ldots\ldots\cos\theta_{n-4}+\ldots\ldots+2\Phi_{n-r+2}\cot\theta_{n-2}\cos\theta_{n-r+1} \\
&\qquad +2\Phi_{n-r+1}\cot\theta_{n-2}+C, \\
&\sum_1^{n-1} X_i^2-2\sin\theta_{n-r}\ldots\ldots\sin\theta_1X_1-\ldots\ldots-2\sin\theta_{n-r}\cos\theta_{n-r+1}X_{n-r} \\
&\qquad -\frac{2\cos\theta_{n-r}}{\sqrt{1-a^2}}X_{n-1}=-2\Phi_{n-r}-2\Phi_{n-r-1}\sin\theta_{n-r}-\ldots\ldots \\
&\qquad -2\Phi_1\sin\theta_{n-r}\ldots\ldots\sin\theta_2+C,
\end{aligned}\right\} \quad (38)$$

the centres of which lie respectively on the hyperboloid of revolution

$$X_{n-r+1}^2+X_{n-r+2}^2+\ldots\ldots+X_{n-2}^2-\frac{X_{n-1}^2}{\dfrac{a^2}{1-a^2}}=-1,\ X_i=0,\ \ i=1,2,\ldots\ldots,n-r, \quad (39)$$

and on the ellipsoid

$$\sum_1^{n-r} X_i^2+\frac{X_{n-1}^2}{\dfrac{1}{1-a^2}}=1, \qquad X_{n-r+1}=0,\ldots\ldots,X_{n-2}=0. \quad (40)$$

These quadrics are focal, the vertex of one passing through the focus of the other. They are, moreover, perpendicular to each other, having the X_{n-1}-axis in common.*

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* The two spaces in which the quadrics are immersed exhibit the maximum of perpendicularity expressed by the fraction $\frac{r-2}{r-1}$; See Schoute, *Mehrdimensionale Geometrie*, 1. Theil, p. 49.

The surfaces (24) may also be considered as the envelope of the two spheres

$$\left.\begin{aligned}
&\text{a)}\ \Sigma X_i^2+4\rho_1 X_1+\ldots\ldots+4\rho_{n-r}X_{n-r}+(1-2\sum_1^{n-r}\rho_i^2)X_{n-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =-4\Phi_1-\ldots\ldots-4\Phi_{n-r}+C-2c,\\
&\text{b)}\ \Sigma X_i^2-4\rho_{n-r+1}X_{n-r+1}-\ldots\ldots-4\rho_{n-2}X_{n-2}-(1-2\sum_{n-r+1}^{n-2}\rho_i^2)X_{n-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =4\Phi_{n-r+1}+\ldots\ldots+4\Phi_{n-2}+C+2c,
\end{aligned}\right\}\quad(41)$$

the centres of which lie on the paraboloids

$$\left.\begin{aligned}
&2X_{n-1}=2-\frac{\sum_1^{n-r}X_i^2}{4},\quad X_i=0;\qquad (i=n-r+1,\ldots,n-2),\\
&2X_{n-1}=2-\frac{\sum_{n-r+1}^{n-2}X_i^2}{4},\quad X_i=0;\qquad (i=1,2,\ldots,n-r).
\end{aligned}\right\}\quad(42)$$

§ 5. *The Generalized Dupin Cyclides in S_{n-1}.*

If in (37) we put $\Phi_1=\Phi_2=\Phi_3=\ldots\ldots=\Phi_{n-r-1}=0$, $\Phi_{n-r}=1-\frac{ka}{\sqrt{1-a^2}}\cos\theta_{n-r}$, $\Phi_{n-r+1}=\Phi_{n-r+2}=\ldots\ldots=\Phi_{n-3}=0$, $\Phi_{n-2}=-\frac{k}{\sqrt{1-a^2}}$, and then derive the envelope of the resulting flat in the usual way we get the surface in the following parametric form:

$$\left.\begin{aligned}
X_1&=\frac{\sin\theta_{n-r}\ldots\sin\theta_2\sin\theta_1(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
X_2&=\frac{\sin\theta_{n-r}\ldots\sin\theta_2\cos\theta_1(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
X_{n-r}&=\frac{\sin\theta_{n-r}\cos\theta_{n-r-1}(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
X_{n-r+1}&=\frac{-\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-4}(a\cos\theta_{n-r}+k\sqrt{1-a^2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
X_{n-2}&=-\frac{\cos\theta_{n-2}\sin\theta_{n-r+1}(a\cos\theta_{n-r}+k\sqrt{1-a^2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
X_{n-1}&=\frac{\sqrt{1-a^2}}{a}\cdot\frac{a\cos\theta_{n-r}+k\sqrt{1-a^2}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}}-\frac{k}{a}.
\end{aligned}\right\}\quad(43)$$

Eliminating the parameters θ_i we obtain the quartic surface

$$(1-a^2)\,[\sum_1^{n-1}X_i^2-k^2-1]^2=4\left[a^2\left(X_{n-1}+\frac{k}{a}\right)^2-(1-a^2)\,(\sum_{n-r+1}^{n-2}X_i^2)\right],\quad(44)$$

which, as we shall prove, is a generalization of the Dupin cyclide of ordinary space. The focal sheets of the surface are given by the formulae

$$\bar{X}_i = X_i + R_k x_i,$$

where R_k are the $n-2$ principal radii of curvature calculated from the equations of Olinde Rodrigue:

$$\frac{\partial X_i}{\partial \theta_k} + R_k \frac{\partial x}{\partial \theta_k}, \qquad \begin{Bmatrix} i=1, 2, \ldots, n-1 \\ k=1, 2, \ldots, n-2 \end{Bmatrix},$$

where x_i have the values given in (29). Although a general surface (37) will have $n-2$ different focal sheets, in the case of the surface (43) it is found that *only two focal sheets exist*, viz.,

$$\left.\begin{aligned} &\text{a)}\ X_1^2 + X_2^2 + \ldots + X_{n-r}^2 + \frac{X_{n-1}^2}{\dfrac{1}{1-a^2}} = 1, \quad X_i = 0, \quad i = n-r+1, \ldots, n-2, \\ &\text{b)}\ X_{n-r+1}^2 + X_{n-r+2}^2 + \ldots + X_{n-2}^2 - \frac{X_{n-1}^2}{\dfrac{a}{1-a^2}} = -1, \quad X_i = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, n-r, \end{aligned}\right\} \quad (45)$$

a pair of focal quadrics, of which one is an ellipsoid immersed in the space $X_{n-r+1} = X_{n-r+2} = \ldots = X_{n-2} = 0$, and the other a hyperboloid immersed in the space $X_1 = X_2 = \ldots = X_{n-r} = 0$. If in (38) we put $\Phi_1 = \Phi_2 = \ldots = \Phi_{n-r-1} = 0$, $\Phi_{n-r} = 1 - \dfrac{ak}{\sqrt{1-a^2}} \cos\theta_{n-r}$, $\Phi_{n-r+1} = \ldots = \Phi_{n-3} = 0$, $\Phi_{n-2} = \dfrac{-k}{\sqrt{1-a^2}}$, and $C = 1 + k^2$, we have the two spheres,

$$\left.\begin{aligned} &\text{a)}\ \sum_1 X_i^2 + 2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}X_{n-r+1} + \ldots \\ &\qquad + 2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{n-2} + \frac{2a\csc\theta_{n-2}}{\sqrt{1-a^2}}X_{n-1} = 1 + k^2 - \frac{2k}{\sqrt{1-a^2}}\csc\theta_{n-2}, \\ &\text{b)}\ \Sigma X_i^2 - 2\sin\theta_{n-r}\ldots\sin\theta_1 X_1 - \ldots - 2\sin\theta_{n-r}\cos\theta_{n-r+1}X_{n-r} \\ &\qquad - \frac{2\cos\theta_{n-r}}{\sqrt{1-a^2}}X_{n-1} = 1 + k^2 - 2\left[1 - \frac{ak}{\sqrt{1-a^2}}\cos\theta_{n-r}\right], \end{aligned}\right\} \quad (46)$$

the radii R_1 and R_2 of which are

$$R_1 = \pm\left[\frac{\csc\theta_{n-2}}{\sqrt{1-a^2}} - k\right], \quad R_2 = \pm\left[\frac{a}{\sqrt{1-a^2}}\cos\theta_{n-r} + k\right],$$

from which we derive

$$R_1 + R_2 = \pm\left[\frac{\cos\theta_{n-2} + a\cos\theta_{n-r}}{\sqrt{1-a^2}}\right] = D. \qquad (47)$$

Hence, *the surface* (43) *is the envelope of two sets of spheres that touch, and whose centres lie respectively on the ellipsoid* (45a) *and hyperboloid* (45b). *The surface is thus a generalization of the Dupin cyclide in ordinary space.* If k varies we get parallel cyclides as appears from (47).

If we give to r in (43) the successive values $3, 4, \ldots, n-1$ we obtain $n-3$ different types of cyclides.* Thus, if $n=3$ the focal hyperboloid becomes a hyperbola lying in the plane $X_1=X_2=\ldots=X_{n-3}=0$, while the ellipsoid is $n-3$ dimensional and is immersed in the space $X_{n-2}=0$. If S_{n-1} is an odd space, and $r=\frac{n+2}{2}$, the focal quadrics are of the same dimension $=\frac{n-2}{2}$, and the equation of the corresponding cyclide is

$$(1-a^2)\,[\Sigma X_i^2-k^2-1]^2 \\ =4\left[a^2\left(X_{n-1}+\frac{k}{a}\right)^2-(1-a^2)\,(X_{\frac{n}{2}}^2+X_{\frac{n+2}{2}}^2+\ldots+X_{n-2}^2)\right]. \quad (48)$$

§ 6. We shall next discuss the type of cyclides for which $a=1$. In (24) we put $\phi_i=\frac{k+2}{4}\rho_i^2$, $i=1, 2, \ldots, n-r$, $\phi_{n-r+k}=\frac{k-2}{4}\rho_{n-r+k}^2$, $k=1, 2, \ldots, r-2$, $c=\frac{k}{4}$, and obtain the surface in the following parametric form:

$$\left.\begin{aligned} &2X_i-2\rho_iX_{n-1}+(k+2)\rho_i=0, \qquad i=1, 2, \ldots, n-r,\\ &2X_{n-r+k}-2\rho_{n-r+k}X_{n-1}+(k-2)\rho_{n-r+k}=0, \qquad k=1, 2, \ldots, r-2,\\ &X_{n-1}=\frac{\frac{k+2}{2}\sum\limits_1^{n-r}\rho_i^2+\frac{k-2}{2}\sum\limits_{n-r+1}^{n-2}\rho_i^2-\frac{k}{2}}{1+\Sigma\rho_i^2}, \end{aligned}\right\} \quad (49)$$

or, in Cartesian form,

$$2X_{n-1}\left(\sum_1^{n-1}X_i^2+\frac{k^2-4}{4}\right)-(k-2)\sum_1^{n-r}X_i^2 \\ -(k+2)\sum_{n-r+1}^{n-2}X_i^2-kX_{n-1}^2+\frac{k(k^2-4)}{4}=0. \quad (49')$$

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* The types for which $r>\frac{n+2}{2}$ do not differ essentially from those for which $r<\frac{n+2}{2}$; the focal ellipsoid and hyperboloid merely interchange. We need therefore consider only the surfaces of type $r\lesseqgtr\frac{n+2}{2}$.

The two sets of spheres (41) are, putting $C=\frac{k^2}{4}$,

$$\left.\begin{aligned}
&\sum_1^{n-1} X_i^2+4\rho_1 X_1+\ldots\ldots+4\rho_{n-r}X_{n-r}+(1-2\sum_1^{n-r}\rho_i^2)X_{n-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =-(k+2)\sum_1^{n-r}\rho_i^2+\frac{k^2}{4}-\frac{k}{2},\\
&\sum_1^{n-1} X_i^2-4\rho_{n-r+1}X_{n-r+1}-\ldots\ldots-4\rho_{n-3}X_{n-3}-(1-2\sum_{n-r+1}^{n-2}\rho_i^2)X_{n-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =(k-2)\sum_{n-r+1}^{n-2}\rho_i^2+\frac{k^2}{4}+\frac{k}{2},
\end{aligned}\right\}\quad(50)$$

and the radii are, respectively:

$$R_1=\pm\tfrac{1}{2}(1+2\sum_1^{n-r}\rho_i^2-k),\quad R_2=\pm\tfrac{1}{2}(1+2\sum_{n-r+1}^{n-2}\rho_i^2+k).\qquad(51)$$

$\therefore R_1+R_2=\pm(1+\sum_1^{n-2}\rho_i^2)=$ Distance between the centres.

Hence, *the cyclide of the third order is the envelope of two sets of spheres that touch, and whose centres lie respectively on two focal paraboloids of revolution whose equations are*

$$\left.\begin{aligned}
&4X_{n-1}+2=\sum_{n-r+1}^{n-2}X_i^2,\quad X_1=X_2=X_{n-r}=0,\\
&4X_{n-1}-2=-\sum_1^{n-r}X_i^2,\quad X_{n-r+1}=X_{n-r+2}=\ldots\ldots=X_{n-2}=0.
\end{aligned}\right\}\quad(52)$$

If $r=3$, one paraboloid is one-dimensional, *i. e.*, a parabola, while the other is of dimension $n-3$. If S_{n-1} is an odd space and $r=\frac{n+2}{2}$, both paraboloids are of the same dimension, viz.: $\frac{n}{2}$. All the cyclides obtained by putting $r=3,\ldots,n=1$ in succession in (49) have the same spherical representation, while in the case of the hyperbolic-elliptic types we get a different orthogonal system of circles on the Gaussian sphere for each value of r.* If k varies we have a system of parallel cyclides as before.

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*Schoute in his *Mehrdimensionale Geometrie*, II. Theil, pp. 316–320, has by a synthetic method borrowed from 3-space, derived the type for which $r=n-1$, *i. e.*, where the focal quadrics are an ellipse and a hyperboloid of revolution of $n-3$ dimensions. That this method leads to the n-dimensional generalization of Dupin's cyclide in the sense that we get *all* such cyclides is, however, not true, although the opposite might be inferred from the author's statement. On p. 320 the author mentions the existence of two families of spheres of the general nature considered above, but does not consider their envelope. The author's reference to a short article in *Verslagen der Akademie von Amsterdam*, vom Februar 1905, I have not been able to look up.

§ 7. *Some Geometrical Properties of the Cyclides in General Space.*

The loci on the cyclide where the radii of the two sets of spheres equal zero will be spreads of double points on the surface. We obtain these point-spheres by equating R_1 and R_2 to zero giving

$$\csc\theta_{n-2}=k\sqrt{1-a^2},\quad \cos\theta_{n-r}=-\frac{k}{a}\sqrt{1-a^2}.$$

Substituting the values of $\cot\theta_{n-2}$ and $\csc\theta_{n-2}$ in (43) and eliminating the remaining parameters θ_i, we get an r—3-dimensional sphere whose equations are

$$X_1=X_2=\ldots\ldots=X_{n-r}=0,\quad \sum_{n-r+1}^{n-2}X_i^2=k^2(1-a^2)-1,\quad X_{n-1}=ka, \tag{53}$$

all the points of which must be considered as centres of null-spheres. Again, substituting the values of $\cos\theta_{n-r}$ and $\sin\theta_{n-r}$ in (43) we get a second locus of double points, viz.:

$$\sum_1^{n-r}X_i^2=\frac{a^2-k^2(1-a^2)}{a^2},\quad X_{n-r+1}=X_{n-r+2}=\ldots\ldots=X_{n-2}=0,\quad X_{n-1}=\frac{k}{a}, \tag{54}$$

(53) lies on the focal hyperboloid H_{r-2} and (54) on the focal ellipsoid E_{n-r}. We shall denote these spheres by Σ_{r-3} and Σ_{n-r-1}, and the two systems of spheres by S_h and S_e, the spheres S_h having their centres on H_{r-2}, and the spheres S_e on E_{n-r}. Since all the point-spheres having their centres on Σ_{r-3} are tangents to all the spheres S_h, their centres must belong to these spheres, hence all the spheres S_h pass through the locus Σ_{r-3}; similarly all spheres S_e pass through the locus Σ_{n-r+1}. We may therefore say that *the quartic cyclides in S_{n-1} is the envelope of spheres having their centres on the ellipsoid E_{n-r} and passing through the sphere-locus Σ_{r-3} on H_{r-2}*; or, *it is the envelope of spheres having their centres on the hyperboloid H_{r-2} and passing through the sphere-locus Σ_{n-r-1} on the ellipsoid E_{n-r}.*

Consider the flat

$$u_1X_1+u_2X_2+\ldots\ldots+u_{n-1}X_{n-1}+p=0; \tag{55}$$

the condition that it shall be tangent to H_{r-2} and E_{n-r} are

$$\sum_1^{n-r}u_i^2+\frac{1}{1-a^2}u_{n-1}^2=p^2,\quad -\sum_{n-r+1}^{n-2}u_i^2+\frac{a^2}{1-a^2}u_{n-1}^2=p^2, \tag{56}$$

from which it follows that we must have $\sum_1^{n-1}u_i^2=0$, which means that *the flat is isotropic.* Introducing in (55) the parameters

$$u_1=2\rho_1,\ \ldots\ldots,\ u_{n-3}=2\rho_{n-3},\quad u_{n-2}=1-\Sigma\rho_i^2,\quad u_{n-1}=i(1+\Sigma\rho_i^2),$$

we obtain the ∞^{n-3} flats

$$2\rho_1 X_1 + 2\rho_2 X_2 + \ldots + (1-\Sigma\rho_i^2) X_{n-2} + i(1+\Sigma\rho_i^2) X_{n-1} + p = 0. \tag{57}$$

where p has the value

$$p = \pm\sqrt{4\sum_1^{n-r}\rho_i^2 - \frac{(1+\Sigma\rho^2)^2}{1-a^2}},$$

The equations of the "developable" surface obtained by letting the ρ's vary are as follows:

$$\left.\begin{aligned} X_i &= \frac{-2i\rho_i}{1+\Sigma\rho^2} X_{n-1} + \frac{2a^2\rho_i}{p(1-a^2)}, \quad (i=1, 2, \ldots, n-r), \\ X_i &= \frac{-2i\rho_i}{1+\Sigma\rho_i^2} X_{n-1} + \frac{2\rho_i}{p(1-a^2)}, \quad (i=n-r+1, \ldots, n-3), \\ X_{n-2} &= -\frac{i(1-\Sigma\rho_i^2)}{1+\Sigma\rho_i^2} X_{n-1} + \frac{1-\Sigma\rho_i^2}{p(1-a^2)}. \end{aligned}\right\} \tag{58}$$

In a former paper* we have studied isotropic complexes of which (58) is a special case. Since the complex contains ∞^{n-3} isotropic lines we shall denote it by the symbol Δ_{n-3}. By a theorem proved in A. I., p. 207, we know that it has a focal surface which is the envelope of the ∞^{n-4} isotropic 2-spreads contained in it, and that this focal surface has in general $n-3$ sheets. The edges of regression on the surface are minimal curves satisfying a certain system of differential equations. In the special case of (58) it may be proved that *there are only three sheets;* in fact, the determinant equation (13), p. 205, l. c., which determines the number of sheets, reduces to the simple form

$$(\sigma-A)^{n-r-1}(\sigma-B)^{r-3}\left[\sigma - A - \frac{16a^2\sum_1^{n-r}\rho_i^2}{p^3(1-a^2)}\right] = 0, \tag{59}$$

where

$$A = \frac{4}{p} - \frac{2(1+\Sigma\rho_i^2)}{p(1-a^2)}, \quad B = \frac{-2(1+\Sigma\rho_i^2)}{p(1-a^2)}.$$

Substituting the values of σ in equations (60), footnote, we obtain the three focal sheets, *two of which are seen to be precisely the focal quadrics* H_{r-3} and

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* A, §§ 1–5 where the formulae developed are true for any space, even or odd, if instead of the parameters α_i and β_i we use ρ_i (see II, § 1, of this paper). The equation determining the focal sheets becomes the equation (26), note, when written of order $n-3$. The equations of the focal sheets take the form:

$$\left.\begin{aligned} X_i &= \frac{\rho_i\sigma_k}{2} - \frac{p}{2}, \quad X_{n-1} = \frac{1-\Sigma\rho^2{}_i}{4}\sigma_k + \frac{1}{2}\Sigma p_i\rho_i - \frac{p}{2}, \\ X_{n-1} &= i\left[\frac{1+\Sigma\rho_i{}^2}{4}\sigma_k - \frac{1}{2}\Sigma p_i\rho_i + \frac{p}{2}\right], \quad \left\{\begin{matrix} i=1, 2, \ldots, n-3 \\ k=1, 2, 3 \end{matrix}\right\}. \end{aligned}\right\} \tag{60}$$

E_{n-r}. The surface of reference of Δ_{n-3} is the imaginary ellipsoid in the space $X_{n-1}=0$:

$$\frac{(1-a^2)}{a^2}\sum_1^{n-r} X_i^2+(1-a^2)\sum_{n-r-1}^{n-2} X_i^2+1=0, \quad X_{n-1}=0 \tag{61}$$

obtained by putting $X_{n-1}=0$ in (58). We have thus proved the

THEOREM. *The tangent $n-2$-flats common to the two focal quadrics H_{r-3} and E_{n-r} generate an isotropic complex Δ_{n-3} which has H_{n-3} and E_{n-r} for two of its three focal sheets and the imaginary quadric* (61) *for surface of reference.*

We shall now return to the study of the quartic cyclide (44). Any straight line $\overline{BA}$ joining a point A on Σ_{r-4} to a point B on Σ_{n-r-1} is an isotropic line. In fact, from (53) and (54) we find that the distance is:

$$D=\sqrt{k^2(1-a^2)-1+\frac{a^2-k^2(1-a^2)}{a^2}+k^2\left(a-\frac{1}{a}\right)^2}=0.$$

The ∞^{n-4} isotropic lines generate a locus D_{n-4} which is the intersection of the quadric cylinders

$$\sum_1^{n-r} X_i^2=\frac{a^2r_2^2}{k^2(1-a^2)^2}[X_{n-1}+ak]^2, \quad \sum_{n-r+1}^{n-2} X_i^2=\frac{a^2r_1^2}{k^2(1-a^2)}\left[X_{n-1}+\frac{k}{a}\right]=0.^* \tag{62}$$

If we substitute the values of $\sum_1^{n-r} X_i^2$ and $\sum_{n-r+1}^{n-2} X_i^2$ from these equations in (44) we find that it is identically satisfied, hence *the ∞^{n-4} isotropic lines $\overline{AB}$ lie on the cyclide.* But they also belong to the isotropic complex Δ_{n-3}, hence

THEOREM. *The quartic cyclides in S_{n-1} is inscribed in the isotropic complex Δ_{n-3}, the locus of contact being the intersection of two quadric cylinders* (62).

Since the lines $\overline{AB}$ are lines of contact between the point-spheres on Σ_{r-4} and Σ_{n-r-1} and the cyclides, they are lines of curvature.

Consider any two points A and B on Σ_{r-3} and Σ_{n-r-1}. The two tangent flats to H_{r-2} and E_{n-r} at A and B respectively intersect the X_{n-1}-axis in a point C whose coordinates are:

$$X_1=X_2=\ldots\ldots=X_{n-2}=0, \quad X_{n-1}=\frac{-a}{k(1-a^2)}.$$

The sphere having its centre at this point and radius equal to $\overline{CB}$ (or $\overline{CA}$) is

$$\Sigma X_i^2+\left[X_{n-1}+\frac{a}{k(1-a^2)}\right]^2=\frac{[1-k^2(1-a^2)][a^2-k^2(1-a^2)]}{k^2(1-a^2)^2}, \tag{63}$$

---

* r_1 and r_2 in these equations are the radii of the spheres (53) and (54).

which, as is easily seen from equations (53) and (54), contains the loci of double points Σ_{r-3} and Σ_{n-r-1}. But the isotropic line $\overline{BA}$ is normal to the radius $\overline{CB}$ (or $\overline{CA}$), and, since it is isotropic, it must lie on the sphere. Hence, the sphere (64) is tangent to the cyclide along the ∞^{n-4} isotropic lines $\overline{AB}$. We shall call this sphere *the principal sphere.**

The point-spheres on Σ_{r-3} intersect the space in which is immersed the focal hyperboloid H_{r-2} in an $r-2$-dimensional sphere K_{r-2} whose equations are:

$$\sum_{n-r+1}^{n-2} X_i^2 + \left(X_{n-1} + \frac{k}{a}\right)^2 = \frac{k^2(1-a^2)-a^2}{a^2}, \quad X_i = 0, \quad i = 1, 2, \ldots, n-r, \quad (64)$$

which is tangent to H_{r-2} along Σ_{r-3}. In the same way, the point-spheres having their centres on Σ_{n-r-1}, intersect the space in which is immersed the focal ellipsoid E_{n-r} in a sphere K_{n-r} of $n-r$ dimensions whose equations are:

$$\sum_{1}^{n-r} X_i^2 + (X_{n-1} + ka)^2 = 1 - k^2(1-a^2), \quad X_i = 0, \quad i = n-r+1, \ldots, n-2, \quad (65)$$

and this sphere is tangent to the focal ellipsoid E_{n-r} along Σ_{n-r-1}. The proof being very simple need not be given here. We shall call the spheres K_{r-2} and K_{n-r} *the focal loci of the cyclide.** *The focal loci are normal to the principal sphere* as may easily be proved from equations (63), (64) and (65).

We have seen that all the spheres S_e having their centres on the focal ellipsoid E_{n-r} pass through Σ_{r-3} and that all the spheres S_h having their centres on H_{r-2} pass through Σ_{n-r-1}. Consider now a cone with vertex at any point A on Σ_{r-3} and passing through the ellipsoid E_{n-r}. This cone is a cone of revolution whose axis is a tangent to the hyperboloid at A, which, as we have seen, passes through the centre C of the principal sphere. In fact, we prove that the radii of the spheres S_e which meet at A, make a constant angle with this tangent. The direction-cosines of any line PA, where P is any point on E_{n-r}, are:

$$\frac{X_1}{PA}, \frac{X_2}{PA}, \ldots, \frac{X_{n-r}}{PA}, \frac{-\bar{X}_{n-r+1}}{PA} \ldots \frac{-\bar{X}_{n-2}}{PA}, \frac{X_{n-1}-\bar{X}_{n-1}}{PA},$$

where $X_1, X_2, \ldots, X_{n-r} X_{n-1}$ are the coordinates of the point P on the ellipsoid, and $\bar{X}_{n-r+1}, \ldots, \bar{X}_{n-2}, \bar{X}_{n-1}$ those of a point on Σ_{n-3}. Again, the direction-cosines of the tangent to the hyperboloid at A are:

$$0, 0, \ldots, 0, \frac{\bar{X}_{n-r+1}}{CA}, \ldots, \frac{\bar{X}_{n-2}}{CA}, \frac{-a\left[k - \frac{1}{k(1-a^2)}\right]}{CA}.$$

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*In adopting the above nomenclature we have followed the analogous one used by Darboux, *Leçons sur les systémes orthogonaux*, p. 489.

The distances $\overline{PA}$ and $\overline{CA}$ are:

$$PA=\frac{a\cos\theta_{n-r}}{\sqrt{1-a^2}}+k,\quad CA=\sqrt{\frac{[1-k^2(1-a^2)][a^2-k(1-a^2)]}{k(1-a^2)}},$$

and the cosine of the angle between these two lines is found to be

$$\cos\omega=\frac{\sqrt{k^2(1-a^2)-1}}{\sqrt{k^2(1-a^2)-a^2}}=\text{const.}$$

In the same way we may prove that the cosine of the angle between the lines $\overline{P'B}$ and $\overline{CB}$ where P' is any point on H_{r-2} and B a point on Σ_{n-r-1}, is constant and equal to

$$\cos\omega'=\frac{\sqrt{a^2-k^2(1-a^2)}}{\sqrt{1-k^2(1-a^2)}}=\sec\omega=\text{const.}$$

and, since $\omega'=\frac{\pi}{2}-\alpha'$, where α' is the angle between the spheres S_h having its centre at P' and the focal locus K_{n-r}, we have

$$\sin\omega'=\cos\alpha'=\frac{i\sqrt{1-a^2}}{\sqrt{k^2(1-a^2)-1}}.$$

Hence, if ω is a real angle, ω' is imaginary and vice versa. We may now state the result in the following

THEOREM. *A general cyclide of Dupin of the fourth order in S_{n-1} is the envelope of spheres whose centres lie on an ellipsoid of revolution E_{n-r} and which intersect a fixed $r-2$-dimensional sphere K_{r-2} at a fixed angle α. It is also the envelope of spheres whose centres lie on a hyperboloid H_{r-2}, and which intersect an $n-r$-dimensional sphere K_{n-r} at a constant angle α'.*

It remains to discuss the nature of the loci of double points on the cyclide. For real cyclides the equation of the principal sphere

$$\sum_1^{n-2}X_i^2+\left[X_{n-1}+\frac{a}{k(1-a^2)}\right]^2=\frac{\left[\frac{1}{1-a^2}-\frac{a^2}{k^2(1-a^2)^2}\right]\left[\frac{a^2}{1-a^2}-\frac{a^2}{k^2(1-a^2)}\right]}{\frac{a^2}{k^2(1-a^2)}}$$

is also real. Let the centre on the X_{n-1}-axis $-\frac{a}{k(1-a^2)}$ be inside the ellipsoid and between the two sheets of the hyperboloid. Σ_{r-3} is then real, the focal locus K_{n-r} is then imaginary and K_{r-2} is real, as is also the principal sphere. The sphere Σ_{n-r} is imaginary and the cyclide has one real locus of double points only.

Let the centre C be between the foci of the hyperboloid and the ellipsoid. The principal sphere is imaginary, since the two factors in the numerator of R^2 have opposite signs. The focal spheres K_{r-2} and K_{n-r} are real since $k^2(1-a^2)-a^2$ is positive. The cyclide has no real double locus.

Finally, when the centre C is outside of the ellipsoid, the principal sphere is again real while Σ_{r-3} is imaginary. The sphere Σ_{n-r-1} is real and the cyclide has one real locus of double points.

§ 8. *Transformations.*

If we transform the cyclide by an inversion whose pole is on one of the focal loci, say K_{r-2}, it is transformed into a torus. In fact, the focal sphere K_{r-2} is transformed into an $r-2$-flat, and the spheres S_h which, as may easily be proved, intersects K_{r-2} orthogonally are transformed into spheres whose centres lie on this flat.* The transform of the cyclide is therefore the envelope of spheres whose centres lie on the $r-2$-flat and touch a given sphere, namely a transform of one of the spheres S_e. By revolving this latter sphere about the flat we get a torus, the transform of the cyclide. If the pole lies on the sphere K_{n-r} this sphere is transformed into an $n-r$-flat and the spheres S, which are orthogonal to K_{n-r}, are transformed into spheres whose centres lie on the $n-r$-flat. These spheres touch any one of the transforms of S_h and the new surface is therefore obtained by revolving the sphere S_h about the $n-r$-flat as an axis. The surface is therefore a torus. *There are ∞^{n-2} inversions which transform a cyclide into a torus.* If, in particular, the pole lies on the sphere Σ_{n-r-1} (or on Σ_{r-3}) we obtain a cone of revolution. There are therefore ∞^{n-r-1} inversions which will transform a cyclide of type $r \leq \frac{n+2}{2}$ into a cone of revolution.

Since the cyclide has only two focal sheets, the determinant in § 3, note, has only two roots, σ_1 and σ_2, of multiplicity $r-2$ and $n-r$, respectively. There exist, therefore, corresponding to $R_1\infty^{r-3}$ principal directions on the surface which lie in a space S_{n-r}; and corresponding to R_2 there are ∞^{n-r-1} principal directions lying in a space S_{r-2}.† Thus we found that from any point A on the locus of double points Σ_{r-3} pass ∞^{n-r-1} isotropic lines forming an isotropic cone of revolution whose elements are lines of curvature. Through any point P not on Σ_{r-3} or Σ_{n-r-1} pass two pencils of ∞^{n-r-1} and ∞^{r-3} circles, respectively. The circles through P generate two spherical spaces S_{n-r} and

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* As a consequence we have a new mode of generation: *a cyclide of the fourth order is generated in two ways by spheres whose centres lie on a hyperboloid of revolution H_{r-2} (or an ellipsoid of revolution E_{n-r}) and which intersect a sphere K_{r-2} (or K_{n-r}) at right angles.*

† See Bianchi, *Lezioni,* Vol. I, p. 369. Seconda edizione.

S_{r-2} which intersect at right angles, as may easily be proved by transforming the cyclide into a cone of revolution.

We consider two special cases. If $k=0$, Σ_{r-3} is the imaginary sphere $\sum_{n-r+1}^{n-2} X_i^2=-1$, $X_1=\ldots=X_{n-r}=X_{n-1}=0$ and Σ_{n-r-1} becomes the unit sphere $\sum_1^{n-r} X_i^2=1$, $X_{n-r+1}=X_{n-r+2}=\ldots=X_{n-1}=0$. The cyclide is symmetrical with respect to all the coordinate planes.

If Σ_{r-3} is the null-sphere whose centre is on the vertex of the focal hyperboloid we have $k=\frac{1}{\sqrt{1-a^2}}$. The focal sphere K_{n-r} becomes the point-sphere and Σ_{n-r-1} an imaginary sphere with radius $\mp\frac{\sqrt{a^2-1}}{a}$ which is immersed in the directrix-space of the focal ellipsoid.

An inversion with pole on Σ_{n-r-1} or Σ_{r-2} will transform the cyclide into a cylinder of revolution. The cyclide will have a single real double locus in finite space as in the general case.

§ 9. *Cyclides of the Third Order.*

If $a=1$ we have a type of cyclides of the third order, the parabolic type, obtained in § 6 (49′). The locus of point-spheres on the surface are gotten by equating to zero the radii of the two sets of spheres (50) which we shall denote by S_{p_1} and S_{p_2}. We have then

$$1+2\sum_1^{n-r}\rho_i^2=k,\quad 1+2\sum_{n-r+1}^{n-2}\rho_i^2=-k. \tag{66}$$

Substituting in equations (50) we have the two point-spheres

$$\left.\begin{aligned}&\sum_1^{n-r-1}(X_i+2\rho_i)^2+\left(X_{n-r}\pm 2\sqrt{\frac{k+1}{2}-\sum_1^{n-r-1}\rho_i^2}\right)^2+\sum_{n-r+1}^{n-2}X_i^2+\left(X_{n-1}+\frac{2-k}{2}\right)^2=0,\\&\sum_1^{n-r}X_i^2+\sum_{n-r+1}^{n-3}(X_i-2\rho_i)^2+\left(X_{n-2}\mp 2\sqrt{\frac{-(1+k)}{2}-\sum_{n-r+1}^{n-2}\rho_i^2}\right)^2\\&\qquad\qquad+\left(X_{n-1}-\frac{2+k}{2}\right)^2=0.\end{aligned}\right\} \tag{67}$$

The loci of point-spheres are therefore:

$$\left.\begin{aligned}&X_1=-2\rho_1,\ \ldots,\ X_{n-r-1}=-2\rho_{n-r-1},\quad X_{n-r}=\mp 2\sqrt{\frac{k+1}{2}-\sum_1^{n-r-1}\rho_i^2},\\&\qquad X_{n-r+1}=\ldots=X_{n-2}=0,\quad X_{n-1}=\frac{k}{2}-1,\\&X_1=\ldots=X_{n-r}=0,\quad X_{n-r+1}=2\rho_{n-r+1}=\ldots=X_{n-3}=2\rho_{n-3},\\&\qquad X_{n-2}=\pm 2\sqrt{\frac{-(1+k)}{2}-\sum_{n-r+1}\rho_i^2},\quad X_{n-1}=1+\frac{k}{2}.\end{aligned}\right\} \tag{68}$$

Eliminating the ρ's we have the spheres

$$\left.\begin{aligned} &\sum_{1}^{n-r} X_i^2 = \quad 2(k-1), \quad X_{n-r+1} = \ldots = X_{n-2} = 0, \quad X_{n-1} = \frac{k}{2} - 1, \\ &\sum_{n-r+1}^{n-2} X_i^2 = -2(k+1), \quad X_1 = \ldots = X_{n-r} = 0, \quad X_{n-1} = \frac{k}{2} + 1, \end{aligned}\right\} (69)$$

which are loci of double points on the surface. If $k>1$ or <-1, one sphere is imaginary while the other is real, and if $k<1$ or >-1, the spheres are both imaginary. We shall denote these spheres by Σ_{p_1} and Σ_{p_2}. All the spheres S_{p_1} pass through Σ_{p_2}, and all spheres S_{p_2} pass through Σ_{p_1}. Hence the cyclides of the third order in S_{n-1} is the envelope of spheres $\begin{Bmatrix} S_{p_1} \\ S_{p_2} \end{Bmatrix}$ having their centres on a paraboloid of revolution $\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$ and passing through a spherical spread $\begin{Bmatrix} \Sigma_{p_2} \\ \Sigma_{p_1} \end{Bmatrix}$ on the paraboloid $\begin{Bmatrix} P_2 \\ P_1 \end{Bmatrix}$. The locus D_{n-4} generated by the isotropic line which joins a point A of Σ_{p_1} to a point B on S_{p_2} belongs to an isotropic developable Δ_{n-3}, and the cyclide is inscribed in this developable along the generators of D_{n-4}, which are lines of curvature of the surface. The focal spheres K_{p_1} and K_{p_2} are

$$\left.\begin{aligned} &\sum_{n-r+1}^{n-2} X_i^2 + [X_{n-1} + \tfrac{1}{2}(2-k)]^2 = 2(1-k), \quad X_1 = X_2 = \ldots = X_{n-r} = 0, \\ &\sum X_i^2 + [X_{n-1} - \tfrac{1}{2}(2+k)] = 2(1+k), \quad X_{n-r+1} = \ldots = X_{n-2} = 0. \end{aligned}\right\} (70)$$

These spheres are tangents to the focal paraboloids P_1 and P_2 along the two loci of double points, respectively. Analogous to the theorem for cyclides of the fourth order we may prove that

A cyclide of the third order in S_{n-1} is in two ways the envelope of spheres $\begin{Bmatrix} S_{p_1} \\ S_{p_2} \end{Bmatrix}$ whose centres lie on a paraboloid of revolution $\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$ and which intersect a fixed sphere $\begin{Bmatrix} K_2 \\ K_1 \end{Bmatrix}$ at a constant angle.

Since the spheres $\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}$ intersect the focal sphere $\begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix}$ at right angles we may say:

A cyclide of the third order is in two ways the envelope of spheres $\begin{Bmatrix} S_{p_1} \\ S_{p_2} \end{Bmatrix}$ whose centres lie on a paraboloid of revolution $\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$ and intersect a fixed sphere $\begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix}$ at right angles.

§ 10. *Surfaces in S_{n-1} Analogous to Moulding Surfaces with Cylindrical Directrix in 3-Space.*

If we put $a=0$ in (29) the spherical representation of the lines of curvature will consist of a system of parallels and meridians on the Gaussian sphere; the corresponding surfaces are then analogous to a certain kind of moulding surfaces in 3-space, viz.: those having a cylindrical directrix. The surface is the envelope of the n—2-flat

$$\sum_1^{n-1} x_i X_i = \Sigma \Phi_i \sqrt{E_i}, \tag{71}$$

the x's and the E's having the following values:

$$\left.\begin{array}{ll} x_1=\sin\theta_{n-2}\sin\theta_{n-r}\ldots\sin\theta_1, & x_2=\sin\theta_{n-2}\sin\theta_{n-r}\ldots\sin\theta_2\cos\theta_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_{n-r}=\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}, & x_{n-r+1}=\cos\theta_{n-r+1}\ldots\cos\theta_{n-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots, \\ x_{n-2}=\cos\theta_{n-2}\sin\theta_{n-r+1}, & x_{n-1}=\cos\theta_{n-r}\sin\theta_{n-2}, \end{array}\right\} \tag{72}$$

$$\left.\begin{array}{ll} E_1=\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_2, & E_2=\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_3, \\ \ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots, \\ E_{n-r}=\sin^2\theta_{n-2}, \quad E_{n-r+1}=\cos^2\theta_{n-2}, & E_{n-r+2}=\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}, \ldots, \\ E_{n-3}=\cos^2\theta_{n-2}\cos^2\theta_{n-4}\ldots\cos^2\theta_{n-r+1}, & E_{n-2}=1. \end{array}\right\} \tag{73}$$

Differentiating (73) partially with respect to the θ's, and solving for the X's, we have the equations of the surface in the form:

$$\left.\begin{array}{l} X_1=\sin\theta_1\ldots\sin\theta_{n-r}\phi_{n-2} \\ \qquad +\sin\theta_1\ldots\sin\theta_{n-r-1}\phi_{n-r}+\ldots+\sin\theta_1\phi_2+\phi_1, \\ X_2=\cos\theta_1\sin\theta_2\ldots\sin\theta_{n-r}\phi_{n-2} \\ \qquad +\cos\theta_1\sin\theta_2\ldots\sin\theta_{n-r-1}\phi_{n-r}+\ldots+\cos\theta_1\phi_2+\psi_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-r}=\cos\theta_{n-r-1}\sin\theta_{n-r}\phi_{n-2}+\cos\theta_{n-r-4}\phi_{n-r}+\psi_{n-r-1}, \\ X_{n-r+1}=\cos\theta_{n-3}\ldots\cos\theta_{n-r+1}\psi_{n-2} \\ \qquad +\cos\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}+\ldots+\cos\theta_{n-3}\psi_{n-4}+\psi_{n-3}, \\ X_{n-r+2}=\sin\theta_{n-3}\cos\theta_{n-4}\ldots\cos\theta_{n-r+1}\psi_{n-2} \\ \qquad +\sin\theta_{n-3}\cos\theta_{n-4}\ldots\cos\theta_{n-r-2}\psi_{n-r+2}+\ldots+\sin\theta_{n-3}\psi_{n-4}+\phi_{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-2}=\sin\theta_{n-r+1}\psi_{n-2}+\phi_{n-r+1}, \quad X_{n-1}=\cos\theta_{n-r}\phi_{n-2}+\psi_{n-r}, \end{array}\right\} \tag{74}$$

where

$$\phi_i=\Phi_i\sin\theta_i+\Phi_i'\cos\theta, \quad \psi_i=\Phi_i\cos\theta_i-\Phi_i'\sin\theta_i.$$

The linear element of the surface is

$$ds^2 = \Sigma b_i d\theta_i,$$

where the b's have the values:

$$\left.\begin{aligned}
b_1 &= [\sin\theta_2 \ldots \sin\theta_{n-r}\phi_{n-2} + \sin\theta_2 \ldots \sin\theta_{n-r-1}\phi_{n-r} \\
&\qquad + \ldots + \phi_2 + \Phi_1 + \Phi_1'']^2, \\
b_2 &= [\sin\theta_3 \ldots \sin\theta_{n-r}\phi_{n-2} + \sin\theta_3 \ldots \sin\theta_{n-r-1}\phi_{n-r} \\
&\qquad + \ldots + \phi_3 + \Phi_2 + \Phi_2'']^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
b_{n-r} &= [\phi_{n-2} + \Phi_{n-r} + \Phi_{n-r}'']^2, \quad b_{n-r+1} = [\psi_{n-2} + \Phi_{n-r+1} + \Phi_{n-r+1}'']^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
b_{n-3} &= [\cos\theta_{n-4} \ldots \cos\theta_{n-r+1}\psi_{n-2} + \cos\theta_{n-4} \ldots \cos\theta_{n-r+2}\psi_{n-r+1} \\
&\qquad + \ldots + \psi_{n-4} + \Phi_{n-3} + \Phi_{n-3}'']^2, \\
b_{n-2} &= [\Phi_{n-2} + \Phi_{n-2}'']^2,
\end{aligned}\right\} \quad (75)$$

from which it is seen that the lines $\theta_1 = c_1$, $\theta_2 = c_2$, ..., $\theta_{n-3} = c_{n-3}$ are geodetic.

The moulding surfaces (74) have certain geometric properties which we shall now discuss.* There are $n-3$ types of these surfaces obtained by giving to r the successive values 3, 4, ..., $n-1$; however, these types are not distinct, in fact, the types for which $r=k$ and $r=n-k+2$ will not yield essentially different surfaces as is seen from equations (72) and (74). If n is odd there are $\frac{n-3}{2}$ distinct types, and $\frac{n-2}{2}$ if n is even.

Consider the surface V_{n-r},

$$\left.\begin{aligned}
\xi_1 &= \sin\theta_1 \ldots \sin\theta_{n-r-1}\phi_{n-r} + \ldots + \sin\theta_1\phi_2 + \phi_1, \\
\xi_2 &= \cos\theta_1 \sin\theta_2 \ldots \sin\theta_{n-r-1}\phi_{n-r} + \ldots + \cos\theta_1\phi_2 + \psi_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\xi_{n-r} &= \cos\theta_{n-r-1}\phi_{n-r} + \psi_{n-r-1}, \quad \xi_{n-1} = \psi_{n-r}, \quad \xi_{n-r+1} = \xi_{n-r+2} = \ldots = \xi_{n-2} = 0,
\end{aligned}\right\} \quad (76)$$

immersed in the space $\xi_{n-r+1} = \ldots = \xi_{n-2} = 0$; it is a moulding surface of type $r' = 3$ in a space S_{n-r+1}. Consider also the cylindrical surface C_{n-r+1} obtained by constructing the normals to the space S_{n-r+1} at every point of V_{n-r}; the direction cosines of these normals to V_{n-r} lying in S_{n-r+1} are:

$$\sin\theta_1 \ldots \sin\theta_{n-r}, \ldots, \cos\theta_{n-r-1}\sin\theta_{n-r}, 0, 0, \ldots, 0, \cos\theta_{n-r},$$

and these are also the direction cosines of a normal to C_{n-r+1} at any point of

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*A special case of the surfaces (74) has been obtained by Umberto Sbrana in an article entitled "I sistemi ciclici nello spazio euclidio ad n dimensione," *Rendiconti del Circolo Matematico di Palermo*, Tomo, XIX, pp. 258–290. The surface is one of type $r=3$, the only distinct type that can exist in 4-space ($n=5$) since $\frac{n-3}{2}=1$ for $n=5$.

the generator passing through the point $\theta_1 \ldots . \theta_{n-r}$. We now construct a surface V_{r-2}:

$$\left.\begin{array}{l} \zeta_{n-r+1} = \cos\theta_{n-3} \ldots . \cos\theta_{n-r+1}\psi_{n-2} + \ldots . + \cos\theta_{n-3}\psi_{n-4} + \psi_{n-3}, \\ \zeta_{n-r+2} = \sin\theta_{n-3} \ldots . \cos\theta_{n-r+1}\psi_{n-2} + \ldots . + \sin\theta_{n-3}\psi_{n-4} + \phi_{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \zeta_{n-2} = \sin\theta_{n-r+1}\psi_{n-2} + \phi_{n-r+1}, \quad \zeta_{n-1} = \phi_{n-2}, \quad \zeta_{n-r} = \ldots . = \zeta_1 = 0, \end{array}\right\} \quad (77)$$

which is a moulding surface in space S_{r-1} of type $r'=3$. This space has the maximum of perpendicularity with the space S_{n-r+1}.

A moulding surface V_{n-2} in S_{n-1} may now be constructed geometrically as follows: In the space S_{n-r+1} of the ambient space S_{n-1} we construct a moulding surface V_{n-r} of type $r'=3$, and a cylindrical surface C_{n-r+1} which is the locus of normals to S_{n-r+1} erected at all points of V_{n-r}. We then construct a moulding surface V_{r-2} in space S_{r-1} which has the maximum of perpendicularity with S_{n-r+1} with a convenient orientation of axes $\zeta_{n-r+1}, \ldots, \zeta_{n-1}$. Let this space take all the ∞^{n-r} positions that are obtained by making a certain one of the given axes coincide with a generator of the cylinder C_{n-r+1}, and the remaining $r-2$-axes with the $r-2$ oriented directions normal to C_{n-r+1} in the point in which the generator meets V_{n-r}. The surface generated will be a moulding surface V_{n-2}. This generation may be described as a "rolling" of S_{r-1} on the cylinder C_{n-r+1}, the points of V_{r-2} in S_{r-1} generating the surface V_{n-2}. A moulding surface in S_{n-1} of type $r=3$ is thus generated by a plane S_2 rolling on a cylinder C_{n-2}, the points of the curve V_1 in the plane generating the surface. Hence it follows that *starting with a moulding surface in* 3-*space* (with cylindrical directrix) *the moulding surfaces of type* $r'=3$ *may be generated in the next higher spaces up to, say,* S_{n-r+1} *and* S_{r-1}, *and finally* V_{n-2} *by "rolling" the space* S_{r-1} *on the cylinder constructed with* V_{n-r} *as a base.* S_{r-1} *must have maximum perpendicularity with* S_{n-r+1}. *The surface generated by the points of* V_{r-2} *is of type* r.

If in (74) we put $\theta_{n-2}=c_{n-2}$ we get two moulding surfaces of type $r'=3$, one immersed in a space $S(X_1, X_2, \ldots, X_{n-r}, X_{n-1})$, and the other in a space $S(X_{n-r+1}, \ldots, X_{n-2})$, both being curved varieties on V_{n-2}. Starting with a moulding surface in 3-space we may construct each one of these by rolling a plane in which is drawn a profile-curve $\xi_i=\phi_i$, $\eta_i=\Psi_i$, over a cylindrical directrix C_i whose base is a moulding surface V_{i-1} of type $r'=3$ in the next lower space; since the profile-curve is plane it will generate a set of plane lines of curvature on V_i. The lines of curvature on V_i remain lines of curvature on V_{i+1} in the next higher space. But the lines of curvature θ_{n-2} are also plane on V_{n-2}, hence they are all plane.

The $n-2$ focal sheets are all distinct; in fact, if we calculate the values of the principal radii we find:

$$R_1=\frac{-\sqrt{b_1}}{\sin\theta_2\ldots\sin\theta_{n-r}\sin\theta_{n-2}},\ \ldots,\quad R_{n-r}=\frac{-\sqrt{b_{n-r}}}{\sin\theta_{n-2}},$$

$$R_{n-r+1}=\frac{-\sqrt{b_{n-r+1}}}{\cos\theta_{n-2}},\ \ldots,\quad R_{n-3}=\frac{-\sqrt{b_{n-3}}}{\cos\theta_{n-r+1}\ldots\cos\theta_{n-4}\cos\theta_{n-2}},$$

$$R_{n-2}=-\sqrt{b_{n-2}}.$$

§ 11. *Surfaces of Revolution.*

If $\Phi_1=\Phi_2=\ldots=\Phi_{n-r}=0$, we get a species of surfaces of revolution in S_{n-1}. If we introduce the arc of the curve $\zeta_{n-2}=\psi_{n-2}$, $\zeta_{n-1}=\phi_{n-2}$ as a new parameter u instead of θ_{n-2}, the equations of the surface may be written

$$X_1=\sin\theta_1\ldots\sin\theta_{n-r}U,\ \ldots,\quad X_{n-r}=\cos\theta_{n-r+1}\sin\theta_{n-r}U,$$

$$X_{n-r+1}=\cos\theta_{n-3}\ldots\cos\theta_{n-r+1}\int_{u_0}^{u}\sqrt{1-U'^2}+\cos\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}$$
$$+\ldots+\cos\theta_{n-3}\psi_{n-4}+\psi_{n-3},$$

$$X_{n-r+2}=\sin\theta_{n-3}\ldots\cos\theta_{n-r+1}\int_{u_0}^{u}\sqrt{1-U'^2}+\sin\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}$$
$$+\ldots+\sin\theta_{n-3}\psi_{n-4}+\phi_{n-3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$X_{n-2}=\sin\theta_{n-r+1}\int_{u_0}^{u_1}\sqrt{1-U'^2}+\phi_{n-r+1},\quad X_{n-1}=\cos\theta_{n-r}U,$$

generated by the moulding surface V (equations 76), turning about the coordinate flat $X_1=X_2=\ldots=X_{n-r}=X_{n-1}=0$ as an axis. Consider the variety $\theta_{n-r+1}=c_{n-r+1}$, $\ldots$, $\theta_{n-2}=c_{n-3}$, $u=c$ on the surface; it is a sphere of constant radius $U(c)$ and its centre lies on the axial space $X_1=\ldots=X_{n-r}=X_{n-1}=0$. Let c be fixed while θ_{n-r+1}, $\ldots$, θ_{n-3} vary; the locus of centres is a moulding surface in $S(X_{n-r+1},\ldots,X_{n-2})$ of type $r'=3$. Any line traced on the ∞^{r-2} spheres $u=c$, where c has any value, is a line of curvature on the surface. Since $R_1=R_1=\ldots=R_{n-r}$ the surface has $r-1$ distinct focal sheets. A surface of revolution of type $r=3$ has $r-1=2$ distinct focal sheets.*

§ 12. *Transformations of the Gaussian Sphere. Orthogonal Systems.*

From equations (37) it follows that given the spherical representation (29) the function $W=\Sigma\Phi_i\sqrt{E_i}$ defines all surfaces whose lines of curvature are plane in all systems. The function W satisfies the following system of differential equations:

$$\frac{\partial^2 W}{\partial\theta_i\partial\theta_k}=\frac{\partial\log\sqrt{E_k}}{\partial\theta_i}\frac{\partial W}{\partial\theta_k}+\frac{\partial\log\sqrt{E_i}}{\partial\theta_k}\frac{\partial W}{\partial\theta_i};\qquad i,k=1,2,\ldots,n-2,\quad i\neq k.\quad (78)$$

*A surface of this kind ($r=3$) in 4-space has been constructed by Sbrana, *loc. cit.*, p. 287.

If now we put

a) $$\lambda = \Delta_1' W + (\theta_{n-1} + W)^2 = \sum \frac{1}{E_i}\left(\frac{\partial W}{\partial \theta_i}\right)^2 + (\theta_{n-1} + W)^2,$$

where θ_{n-1} denotes an arbitrary constant, we have the following application of a general theorem proved by Sbrana: *

There exists a transformation of the Gaussian sphere into itself by means of which the differential form (30) *is transformed into one of the form*

$$d\sigma^2 = \lambda^2 \sum_1^{n-2} \left[\frac{\dfrac{\partial\left(\dfrac{W+\theta_{n-1}}{\lambda}\right)}{\partial \theta_i}}{\dfrac{\partial W}{\partial \theta_i}} \right]^2 E_i d\theta_i^2, \quad (\theta_{n-1} = \text{arbitrary constant.})$$

To a point x_i corresponds a point x_i' by means of the equations

b) $$x_i' = x_i - \rho \frac{\theta_{n-1} + W}{\Psi} \left[(\theta_{n-1} + W) x_i + \rho \nabla'(x_i, W) \right],$$

where

c) $$\Psi = \int \sum R_i \frac{\partial W}{\partial \theta_i} d\theta_i \text{ and } -\frac{2\Psi}{\rho} + \Delta' W + (\theta_{n-1} + W)^2 = 0.$$

The corresponding $n-1$-tuple orthogonal system has for linear element

$$ds^2 = \Sigma d\xi_i^2 = \rho^2 d\theta_{n-1}^2 + \lambda^2 \sum_1^{n-2} \left[\frac{\dfrac{\partial\left(\dfrac{W+\theta_{n-1}}{\lambda}\right)}{\partial \theta_i}}{\dfrac{\partial W}{\partial \theta_i}} \right]^2 E_i d\theta_i^2,$$

the coordinates ξ_i being given by the equations:

$$\xi_i = X_i + \rho(\theta_{n-1} + W) x + \rho \nabla'(x_i, W),$$

the symbol ∇' in (b) *being equivalent to the summation*

$$\sum \frac{1}{E_i} \frac{\partial x_i'}{\partial \theta_i} \frac{\partial W}{\partial \theta_i}.$$

All the cyclic systems normal to the surfaces whose tangential equation is given by (37) may be obtained by quadrature. In particular, the cyclic systems normal to the surfaces (74) are gotten by integrating the equation

$$d\Psi = \sum_1^{n-2} R_i \frac{\partial W}{\partial \theta_i} d\theta_i,$$

* Sbrana, *loc. cit.*, p. 227. An application to a very special case is given on p. 282.

where the R's have the values given at the end of § 10 and $W=\Sigma\Phi_i\sqrt{E_i}$, (eq. 71). For the general surface (37) the calculation of the R's involves considerable work.

II.

§ 1. *Transformations. Asymptotic Lines.*

Consider the $n-2$-spread (6),

$$\left.\begin{aligned} X_1 &= \frac{(\alpha_1+\beta_1)}{2}X_{n-1}-\frac{1}{2}(F_{\alpha_1}+F_{\beta_1}),\\ X_2 &= i\frac{(\alpha_1-\beta_1)}{2}X_{n-1}+\frac{i}{2}(F_{\alpha_1}-F_{\beta_1}),\\ &\dots\dots\dots\dots\dots\dots,\\ &\dots\dots\dots\dots\dots\dots,\\ X_{n-1} &= \frac{\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}-F}{1+\Sigma\alpha_i\beta_i}. \end{aligned}\right\} \quad (79)$$

If, as was done in our former paper (A, p. 203) we consider F, α_i, β_i, F'_{α_i}, F'_{β_i} as the surface-elements of a space S_{n-1}, the above system may be looked upon as a contact-transformation which carries the surface-elements of S_{n-1} into those of $\bar{S}_{n-1}$. To the lines of curvature on a surface in S_{n-1} correspond a set of conjugate lines on the transform to which we have given the name Euler's lines or E-lines. These lines have the property of being transformed into asymptotic lines on a surface in a space $\bar{\bar{S}}_{n-1}$ by means of Euler's transformation

$$x_i=F'_{\alpha_i},\quad y_i=\beta_i,\quad p_i=-\alpha_i,\quad q_i=F'_{\beta_i},\quad z=F-\Sigma\alpha_iF'_{\alpha_i}. \tag{80}$$

These E-lines, while they have been implicitly used by several authors,* have never been considered from the view-point of the theory of contact-transformations before the author's treatment of them in an article in *American Transactions*, Vol. VI, pp. 450–471. Since Euler's transformation applies to odd space the E-lines do not exist on a surface in even-dimensional space, and, as a consequence, the asymptotic lines exist in odd space only.

The tangential equation of a surface in an even or odd space S_{n-1} may be written as before:

$$2y_1X_1+2y_2X_2+\dots+(1-\Sigma y_i^2)X_{n-1}+F=0, \tag{I}$$

and, if S_{n-1} is an odd space,

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\dots+(1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0, \tag{II}$$

* Darboux, *Leçons*, Vol. I, pp. 200–202, deuxième édition, Vol. IV, p. 171.

which latter form is reducible to the first by means of the transformation

$$\alpha_1+\beta_1=2y_1, \quad i(\alpha_1-\beta_1)=2y_2, \ldots, \quad (1-\Sigma\alpha_i\beta_i)=1-\Sigma y_i^2.$$

The lines of curvature of the surface M_{n-2} defined as the envelope of the tangent planes (II) are given by the system of differential equations (A, p. 211),

$$\frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\alpha k}}{dF_{\beta_{\frac{n-2}{2}}}}, \quad \frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\beta i}}{dF_{\beta_{\frac{n-2}{2}}}}, \qquad \begin{array}{l} i=1, 2, \ldots, \frac{n-4}{2}, \\ k=1, 2, \ldots, \frac{n-2}{2}, \end{array}$$

to which in the case of (I) corresponds the system

$$\frac{dy_i}{dy_{n-2}}=\frac{dF'_{y_i}}{dF_{n-2}}, \qquad i=1, 2, \ldots, n-3, \tag{81}$$

and we have seen (§ 2) that if the lines of curvature on a surface are *coordinate lines*, the *y's*, considered as functions of the ρ's, must form a completely orthogonal system in an $n-2$-space, and that F is a particular or general solution of the system of differential equations (12). The contact-transformation corresponding to (6) now takes the simple form:

$$X_i=\frac{y_i(\Sigma y_iF'_{y_i}-F)}{1+\Sigma y_i^2}-\tfrac{1}{2}F'_{y_i}, \quad X_{n-1}=\frac{\Sigma y_iF'_{y_i}-F}{1+\Sigma y_i^2}, \tag{82}$$

which transforms spheres into paraboloids of the form:

$$F=a\Sigma y_i^2+\Sigma b_iy_i+c.$$

The Theorem II, (§ 2), is therefore true in any space, and the surfaces obtained by the application of the theory exist in any space, even or odd.

If we consider an odd space $\bar{S}_{n-1}$ and a spread $F=F(\alpha_i, \beta_i)$, Theorem II may be stated thus:

If, on a surface in $\bar{S}_{n-1}$, a system of curves are coordinate E-lines, the coordinates F, α_i, β_i satisfy the system of differential equations

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}-\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}=0, \tag{12a}$$

and α_i, β_i satisfy the $\frac{n-2\cdot n-3}{2}$ relations

$$\sum\left(\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right)=0, \qquad k, k'=1, 2, \ldots, n-2,$$

and the surface F is found by taking any particular or general solution of (12a).

If we transform to $\bar{\bar{S}}_{n-1}$, using Euler's transformation, we have the following

THEOREM III. *If on a surface in $\bar{\bar{S}}_{n-1}$ a system of curves are coordinate asymptotic lines, the coordinates y_i and p_i of the surface-elements on the surface must satisfy the differential equations* (12a) *and also the* $\frac{n-2 \cdot n-3}{2}$ *relations*

$$\sum\left(\frac{\partial y_i}{\partial \rho_{k'}}\frac{\partial p_i}{\partial \rho_k}+\frac{\partial y_i}{\partial \rho_k}\frac{\partial p_i}{\partial \rho_{k'}}\right)=0.$$

This theorem solves the problem of *Lelieuvre* for general odd space. A system of solutions y_i, p_i having been found, q_i and x_i may be found by quadratures and hence also z. It should be observed that $z-\Sigma x_i p_i$ is a particular solution of (12a).

In order to find surfaces with coordinate asymptotic lines it will be most convenient to find in S_{n-1} corresponding surfaces with coordinate lines of curvature. The chief difficulty which we encounter is the fact, pointed out before, that to real elements in S_{n-1} correspond in general imaginary elements in $\bar{\bar{S}}_{n-1}$ with the obvious result that the surface, if real in S_{n-1}, has often an imaginary transform in $\bar{\bar{S}}_{n-1}$ and vice versa. If, however, we have a real surface in $\bar{S}_{n-1}$, *i. e.*, if F considered as a function of α_i, β_i, is real, the corresponding surface in $\bar{\bar{S}}_{n-1}$ is real.

§ 2. Consider the surfaces for which

$$F=2\phi_1(\rho_1)+2\phi_2(\rho_2)+\ldots\ldots+2\phi_{n-2}(\rho_{n-2})+2C$$

obtained on p. 7. If we transform by Euler's transformation we get the surface

$$\left.\begin{aligned} x_i&=\phi'_{2i-1}+i\phi'_{2i}, \quad y_i=\rho_{2i-1}+i\rho_{2i},\\ z&=2\Sigma\phi_i(\rho_i)-\Sigma(\rho_{2i-1}-i\rho_{2i})(\phi'_{2i-1}+i\phi'_{2i})+2C,\end{aligned}\right\} \quad (83)$$

which is real if ϕ_{2i} is an even function of ρ_{2i}.

As a particular case let us take a cyclide of the third order and type $r=\frac{n-2}{2}$ obtained by giving to the ϕ's the following values:*

$$\phi_{2i-1}=\frac{k+2}{2}\rho^2_{2i-1}, \quad \phi_{2i}=\frac{k-2}{2}\rho^2_{2i}, \quad 2C=\frac{k}{2}.$$

We get the real surface

$$x_i=\frac{k+2}{2}\rho_{2i-1}+i\frac{k-2}{2}\rho_{2i}, \quad y_i=\rho_{2i-1}+i\rho_{2i}, \quad z=2i\Sigma\rho_{2i-1}\rho_{2i}+\frac{k}{2}, \qquad (84)$$

that is, the paraboloid

$$z+\frac{1}{2}\sum_1^{\frac{n-2}{2}}\left[x_i^2+\frac{k^2-4}{4}y_i^2-kx_iy_i\right]=\frac{k}{2}. \qquad (84')$$

* This surface differs from (49′) only in orientation of axes.

This surface is generated in two different ways by self-dual flats corresponding to the two sets of curvature-spheres which generate its transform in S_{n-1}. These flats are evidently:

$$\left.\begin{aligned} x_i &= \frac{k-2}{2}y_i + 2\rho_{2i-1}, & x_i &= \frac{k+2}{2}y_i - 2i\rho_{2i}, \\ z &= 2\Sigma[\rho_{2i-1}y_i - \rho_{2i-1}^2] + \frac{k}{2}, & z &= 2i\Sigma[\rho_{2i}y_i - i\rho_{2i}^2] + \frac{k}{2}. \end{aligned}\right\} \quad (85)$$

If we introduce the parameters α_i, β_i instead of the ρ's in these equations we see that they are real flats. The asymptotic lines are to a certain extent indeterminate. In fact, the flats I and II lie entirely in the surface (84′), and through any point on the surface pass two flats so that any curve immersed in either one of them is an asymptotic curve. Through any point will pass $\infty^{\frac{n-4}{2}}$ asymptotic directions.

To the flat

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d, \tag{86}$$

corresponds a sphere (A, p. 217)

$$\left(X_1 + \frac{b_1+c_1}{2}\right)^2 + \left(X_2 - i\frac{b_1-c_1}{2}\right)^2 + \ldots + \left(X_{n-1} - \frac{a-d}{2}\right)^2 = \left(\frac{a+d}{2}\right)^2,$$

but to a sphere corresponds two flats since $R = \pm\frac{a+d}{2}$. We shall call the flat for which $R = +\frac{a+d}{2}$ the *positive correspondent* and the one for which $R = -\frac{a+d}{2}$ the *negative correspondent.* To point-spheres correspond flats for which $a = -d$, that is, the two correspondents coincide. Any pair of + and — correspondents in the same set are said to be *conjugate* with respect to the flat-complex $a+d=0$ (A, p. 217). Comparing now equations (51), p. 20, with (85) and (86), it appears that the flats of the first set are negative correspondents, and those of the second set are positive correspondents. To the nodal loci on the cyclide correspond two sets of flats on the paraboloid (84) belonging to the flat-complexes $a+d=0=\frac{k-2}{2}-2\Sigma\rho_{2i-1}^2+\frac{k}{2}$, and $a+d=0=\frac{k+2}{2}+2\Sigma\rho_{2i}^2+\frac{k}{2}$:

$$\left.\begin{aligned} x_i &= \frac{k-2}{2}y_i + 2\rho_{2i-1}, & x &= \frac{k+2}{2}y_i - 2i\rho_{2i}, \\ z &= 2\Sigma\rho_{2i-1}y_i - \frac{k}{2} + 1, & z &= 2\Sigma\rho_{2i}y_i - \frac{k}{2} - 1, \\ 2\Sigma\rho_{2i-1}^2 &= (k-1), & 2\Sigma\rho_{2i}^2 &= -(1+k). \end{aligned}\right\} \quad (87)$$

They are the common generators of the quadric and the two cylinders,

$$\Sigma\left(x_i-\frac{k-2}{2}y_i\right)^2=2(k-1),\quad \Sigma\left(x_i-\frac{k+2}{2}y_i\right)^2=-2(1+k). \tag{88}$$

If the cyclide is of type $r\neq\frac{n-2}{2}$, we give to the ϕ's the following values:

$$\phi_{2i-1}=\frac{k+2}{4}\rho_{2i-1}^2,\quad \phi_{2i}=\frac{k-2}{4}\rho_{2i}^2,\quad \phi_{2r+t}=\frac{k-2}{4}\rho_{2r+t}^2,\qquad \begin{matrix} i=1,2,\ldots,r,\\ t=1,2,\ldots,r.\end{matrix}$$

The transform of the cyclide is:

$$\left.\begin{aligned} x_j &= \frac{k+2}{2}\rho_{2j-1}+i\frac{k-2}{2}\rho_{2j}, & y_i&=\rho_{2i-1}+i\rho_{2i}, & j&=1,2,\ldots,r,\\ & & & & i&=1,2,\ldots,\frac{n-2}{2},\\ x_{r+s}&=\frac{k-2}{2}[\rho_{2(r+s)-1}+i\rho_{2(r+s)}]y_{r+s}, & z&=i\Sigma\rho_{2j}\rho_{2j-1}+\frac{k}{2}, & & \\ & & & & s&=1,2,\ldots,\frac{n-2-2r}{2}.\end{aligned}\right\} \tag{89}$$

Eliminating the ρ's we get the parabolic surface of $\frac{n-2}{2}+r$ dimensions

$$z=-\frac{1}{2}\sum_1^r\left[x_j^2+\frac{k^2-4}{4}y_j^2-kx_jy_j\right]+\frac{k}{2};\quad x_{r+s}=\frac{k-2}{2}y_{r+s},$$
$$s=1,2,\ldots,\frac{n-2}{2}-r. \tag{90}$$

As in the preceding special case the two sets of spheres generating the cyclide are transformed into two sets of flats:

$$\mathrm{I}\begin{cases} x_j &= \frac{k-2}{2}y_i+2\rho_{2j-1},\\ x_{r+s}&=\frac{k-2}{2}y_{r+s},\\ z &= 2\sum_1^r\rho_{2j-1}y_j-2\sum_1^r\rho_{2j-1}^2+\frac{k}{2}.\end{cases}$$

$$\mathrm{II}\begin{cases} x_j &= \frac{k+2}{2}y_j-2i\rho_{2j},\\ x_{r+s}&=\frac{k+2}{2}y_{r+s}-2(\rho_{2(r+s)-1}+i\rho_{2(r+s)}),\\ z &= 2i\Sigma\rho_{2j}y_j-2\Sigma(\rho_{2(r+s)-1}-i\rho_{2(r+s)})y_{r+s}+2\sum_1^{\frac{n-2}{2}}\rho_{2j}^2+2\sum_{r+1}^{\frac{n-2}{2}}\rho_{2j-1}^2+\frac{k}{2},\\ & \left(j=1,2,\ldots,r,\quad s=r+1,\ldots,\frac{n-2}{2}-r\right).\end{cases}$$

The flats of the first set lie entirely in the surface and may be considered as generating the surface, the Cartesian equation of which is obtained by eliminating the parameters ρ_{2j}. The flats of the second set intersect those of the first in points on the surface, hence the parametric equation (89) may be gotten by solving I and II for x_i, y_i, and z.

Since to every sphere of radius $R \neq 0$ correspond two flats which do not intersect, to every surface in S_{n-1} correspond two surfaces in $\bar{\bar{S}}_{n-1}$; these are said to be conjugate to each other.* The conjugate of the surface (84) is obtained by changing the sign of R_1 and R_2 so that we have

$$\frac{a+d}{2}=R_1=\frac{1}{2}\,[2\Sigma\rho_{2i-1}^2+1-k],\quad R_2=-\frac{1}{2}\,[2\Sigma\rho_{2i}^2+1+k],$$

$$\frac{a-d}{2}=-\frac{1-2\Sigma\rho_{2i-1}^2}{2},\quad \frac{a-d}{2}=\frac{1-2\Sigma\rho_{2i}^2}{2},$$

from which we have

$$a=2\Sigma\rho_{2i-1}^2-\frac{k}{2},\quad d=1-\frac{k}{2};\quad a=-2\Sigma\rho_{2i}^2-\frac{k}{2},\quad d=-\frac{k+2}{2},$$

the conjugate surface is therefore generated by either one of the two sets of flats

$$\left.\begin{aligned} x_i&=\left(2\Sigma\rho_{2i-1}^2-\frac{k}{2}\right)y_i+2\rho_{2i-1}, & x_i&=-\left(2\Sigma\rho_{2i}^2+\frac{k}{2}\right)y_i-2i\rho_{2i},\\ z&=2\Sigma\rho_{2i-1}y_i+1-\frac{k}{2}, & z&=2i\Sigma\rho_{2i}y_i-\frac{k+2}{2}. \end{aligned}\right\}\quad(91)$$

Eliminating the ρ's we have the quartic surface

$$\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2-2\sum_1^{\frac{n-2}{2}}x_iy_i-k\Sigma y_i^2+z(z+k)+\frac{k^2-4}{4}=0,\qquad (i<k),\quad (84'')$$

which intersects the quadric (84′), *i. e.*, its conjugate, along the null-flats of the first and second sets, that is, flats for which

$$2\Sigma\rho_{2i-1}^2+1-k=0 \text{ and } 2\Sigma\rho_{2i}^2+1+k=0,$$

(equations (87) and (88)).

If the cyclide is of type $r \neq \frac{n-2}{2}$ the equations of the surface conjugate to (90) are

$$\left.\begin{aligned} &\sum_1^r(x_iy_k-x_ky_i)^2-2\sum_1^r x_iy_i-k\sum_1^r y_i^2+z(z+k)+\frac{k^2-4}{4}=0,\\ &\sum_1^r(y_ix_{r+s}-x_ky_{r+s})y_i+zy_{r+s}+\frac{k-2}{2}y_{r+s}=0,\quad s=1,2,\ldots,\frac{n-2}{2}-r. \end{aligned}\right\}\quad(90')$$

*Also "reciprocal" in the terminology of S. Lie.

Since the flats of the second set do not lie on the surface (90) the null-flats of the second set will not lie on the intersection of (90) and (90′); only if $r=\frac{n-2}{2}$ will this be true.

§ 3. *Transformation of the Cyclides of the Fourth Order.*

We write the two sets of spheres which generate the cyclide of type $r=\frac{n+2}{2}$ as follows:*

$$\left.\begin{aligned}
&\Sigma X_i^2+2\cot\theta_{n-2}\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-3}X_2+\ldots+2\cot\theta_{n-2}\sin\theta_{\frac{n}{2}}\cdot X_{n-2}\\
&\qquad\qquad+\frac{2a_0\csc\theta_{n-2}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2-\frac{2k}{\sqrt{1-a_0^2}}\csc\theta_{n-2},\\
&\Sigma X_i^2-2\sin\theta_{\frac{n-2}{2}}\ldots\sin\theta_1X_1-\ldots-2\sin\theta_{\frac{n-2}{2}}\cos\theta_{\frac{n-4}{2}}X_{n-3}\\
&\qquad\qquad-\frac{2\cos\theta_{\frac{n-2}{2}}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2-2\left[1-\frac{a_0k}{\sqrt{1-a_0^2}}\cos\theta_{\frac{n-2}{2}}\right].
\end{aligned}\right\}\quad(92)$$

Transforming we get two sets of flats,

$$\mathrm{I(a)}\begin{cases}x_i=ay_i+b_i,\\ z=-\sum\limits_1^{\frac{n-2}{2}}b_iy_i+d,\end{cases}\qquad \mathrm{II(a)}\begin{cases}x_i=\bar{a}y_i+\bar{b}_i,\\ z=\sum\limits_1^{\frac{n-2}{2}}\bar{b}_iy_i+\bar{d},\end{cases}$$

where a, b_i, d, $\bar{a}$, $\bar{b}_i$, $\bar{d}$ have the following values:

$$a=-\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}+k,\qquad \bar{a}=\sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{\frac{n-2}{2}}+k,$$

$$d=-\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}+k,\qquad \bar{d}=-\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{\frac{n-2}{2}}+k,$$

$$b_i=\quad i\cot\theta_{n-2}\phi_i,\qquad \bar{b}_i=-\sin\theta_{\frac{n-2}{2}}\psi_i,$$

ϕ_i and ψ_i having the values:

$$\phi_1=\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-4}\cos\theta_{n-3},\qquad \phi_{i+1}=\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-(i+3)}\sin\theta_{n-(i+2)},$$

$$\psi_1=\sin\theta_{\frac{n-4}{2}}\ldots\sin\theta_1,\qquad \psi_{i+1}=\sin\theta_{\frac{n-4}{2}}\ldots\sin\theta_i\cos\theta_{i-1},$$

$$\left(i=1,2,\ldots,\frac{n-4}{2}\right).$$

---

* These equations differ from (46), only in orientation of the axis $X_1,\ldots,X_{n-2}$; we also have put $a=a_0$ in order to avoid confusing it with the parameter a in I(a) and II(a).

Eliminating the θ's from I (or II) we obtain the quartic surface:

$$(1+a_0)\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2+(1-a_0)\sum_1^{\frac{n-2}{2}}(x_i-ky_i)^2-(1+a_0)\sum_1^{\frac{n-2}{2}}y_i^2$$
$$+(1+a_0)(z-k)-1+a_0=0, \qquad i<k. \quad (93)$$

The conjugate surface is:

$$(1-a_0)\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2+(1+a_0)\sum_1^{\frac{n-2}{2}}(x_i+ky_i)^2-(1-a_0)\sum_1^{\frac{n-2}{2}}y_i^2$$
$$+(1-a_0)(z+k)^2-(1+a_0)=0, \quad (93')$$

generated by the two sets of flats:

$$\text{I(b)}\ \begin{cases} x_i = a'y_i+b_i, \\ z=-\Sigma b_iy_i+d', \end{cases} \qquad \text{II(b)}\ \begin{cases} x_i=\bar{a}'y_i+\bar{b}_i, \\ z=\Sigma\bar{b}_iy_i+\bar{d}', \end{cases}$$

where a', d', $\bar{a}'$, $\bar{d}'$ have the values

$$a'=\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}-k, \qquad \bar{a}'=\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{\frac{n-2}{2}}-k,$$
$$d'=\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}-k, \qquad \bar{d}'=-\sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{\frac{n-2}{2}}-k.$$

These two surfaces intersect along the two sets of null-flats whose equations are:

$$x_i=-a_0ky_i+i\sqrt{1+k^2(1-a_0^2)}\,\phi_i, \qquad x_i=-\frac{k}{a_0}y_i-\frac{\sqrt{a^2-k^2(1-a_0^2)}}{a_0}\psi_i,$$
$$z=-\sum_1^{\frac{n-2}{2}}i\sqrt{1+k^2(1-a_0^2)}\,\phi_iy_i+ak, \qquad z=-\sum\frac{\sqrt{a_0^2-k^2(1-a_0^2)}}{a_0}\psi_iy_i+\frac{k}{a_0}.$$

These flats are the intersections of the quadric cylinders

$$\Sigma(x_i+a_0ky_i)^2=-[1+k^2(1-a_0^2)], \text{ and } \Sigma\left(x_i+\frac{k}{a_0}y_i\right)^2=\frac{a_0^2+k^2(1-a_0^2)}{a_0^2}$$

with the surface (93) or (93').

If the cyclide is of type $r<\frac{n+2}{2}$ we write the two sets of generating spheres as follows:

$$\left.\begin{aligned}
&\Sigma X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}X_2+\ldots \\
&\quad+2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{2(r-2)}+\frac{2a_0\csc\theta_{n-2}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2 \\
&\quad-\frac{2k}{\sqrt{1-a_0^2}}\csc\theta_{n-2}, \\
&\Sigma X_i^2-2\sin\theta_{n-r}\ldots\sin\theta_1X_1-2\sin\theta_{n-r}\ldots\cos\theta_1X_3-\ldots \\
&\quad-2\sin\theta_{n-r}\ldots\cos\theta_{r-3}X_{2(r-2)-1}-2\sin\theta_{n-r}\ldots\cos\theta_{r-2}X_{2(r-2)+1} \\
&\quad-\ldots-2\sin\theta_{n-r}\cos\theta_{n-r-1}X_{n-3}-\frac{2\cos\theta_{n-r}}{\sqrt{1-a_0^2}}X_{n-1} \\
&\quad=1+k^2-2\left[1-\frac{a_0k}{\sqrt{1-a_0^2}}\cos\theta_{n-r}\right].
\end{aligned}\right\} \quad (94)$$

The parameters b_i, c_i, a, d, b_i, $\bar{c}_i$, $\bar{a}$, $\bar{d}$ of the corresponding flats are therefore:

$$\left.\begin{aligned}
&b_1 = i\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3},\\
&b_2 = i\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\sin\theta_{n-3},\ \ldots,\\
&b_{r-2} = i\cot\theta_{n-2}\sin\theta_{n-r+1},\\
&a = -\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}+k,\quad c_i=-b_i,\qquad i=1,2,\ldots,r-2,\\
&d = -\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}+k,\quad b_{r-2+s}=c_{r-2+s}=0,\\
&\qquad\qquad s=1,2,\ldots,\frac{n+2}{2}-r,\\
&\bar{b}_1 = -\sin\theta_{n-r}\ldots\sin\theta_1,\quad \bar{b}_2=\sin\theta_{n-r}\ldots\cos\theta_1,\ \ldots,\\
&\bar{b}_{r-2} = -\sin\theta_{n-r}\ldots\cos\theta_{r-3},\quad \bar{c}_i=\bar{b}_i,\qquad i=1,2,\ldots,r-2,\\
&\bar{b}_{r-2+s} = -\sin\theta_{n-r}\ldots\sin\theta_{r-2+2s}(\sin\theta_{r-3+2s}\cos\theta_{r-4+2s}+i\cos\theta_{r-3+2s}),\\
&\bar{c}_{r-2+s} = -\sin\theta_{n-r}\ldots\sin\theta_{r-2+2s}(\sin\theta_{r-3+2s}\cos\theta_{r-4+2s}-i\cos\theta_{r-3+2s}),\\
&\bar{a} = \sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{n-r}+k,\quad \bar{d}=-\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{n-r}+k,\\
&\qquad\qquad s=1,2,\ldots,\frac{n+2}{2}-r.
\end{aligned}\right\}\quad(95)$$

The two sets of flats may then be written:

$$\mathrm{I(c)}\quad\left\{\begin{aligned}
x_i &= ay_i+b_i,\\
x_{r+s-2} &= ay_{r+s-2},\\
z &= -\sum_1^{r-2} b_i y_i + d,
\end{aligned}\right.$$

$$\mathrm{II(c)}\quad\left\{\begin{aligned}
x_i &= \bar{a}y_i+\bar{b}_i\\
x_{r+s-2} &= \bar{a}y_{r+s-2}+\bar{b}_{r+s-2},\\
z &= \sum_1^{r-2}\bar{b}_i y_i+\Sigma\bar{c}_{r+s-2}y_{r+s-2}+\bar{d},
\end{aligned}\right.\qquad \left\{\begin{aligned}&i=1,2,\ldots,r-2,\\&s=1,2,\ldots,\frac{n+2}{2}-r\end{aligned}\right\}.$$

Eliminating the parameters θ_i from I(c) we obtain a surface of $\frac{n-6}{2}+r$ dimensions whose equations are

$$\left.\begin{aligned}
&(1+a_0)\sum_1^{r-2}(x_iy_k-x_ky_i)^2+(1-a_0)\sum_1^{r-2}(x_i-ky_i)^2-(1+a_0)\sum_1^{r-2}y_i^2\\
&\qquad\qquad+(1+a_0)(z-k)^2-(1-a_0)=0,\\
&(1+a_0)\sum_1^{r-2}(x_{r+s-2}y_i-x_iy_{r+s-2})y_i-(1+a_0)(z-k)y_{r+s-2}\\
&\qquad+(1-a_0)(x_{r+s-2}-ky_{r+s-2})=0,\qquad s=1,2,\ldots,\frac{n+2}{2}-r.
\end{aligned}\right\}(96a)$$

Its conjugate, which may be obtained by changing the sign of a_0 and k in the above equations, has for equations:

$$\left.\begin{aligned}
&(1-a_0)\sum_1^{r-2}(x_iy_k-x_ky_i)^2+(1+a_0)\sum_1^{r-2}(x_i+ky_i)^2-(1-a_0)\sum_1^{r-2}y_i^2\\
&\qquad\qquad+(1-a_0)(z+k)^2-(1+a_0)=0,\\
&(1-a_0)\sum_1^{r-2}(x_{r+s}y_i-x_iy_{r+s-2})y_i-(1-a_0)(z+k)y_{r+s-2}\\
&\qquad\qquad+(1+a_0)(x_{r+s-2}+ky_{r+s-2})=0.
\end{aligned}\right\}(96b)$$

The second set of flats which do not lie on the surface intersect those of the first in a point on the surface; the parametric equations of the surface will therefore be found by solving I(c) and II(c) for x_i, y_i, and z. The null-flats of I(c), which also lie on the surface, form part of the intersection of the surface (96a, b) with its conjugate. The second set of null-flats are common tangent flats of the two conjugate surfaces along the remaining locus of intersection in the finite part of space. At infinity there is a common cone of intersection whose equations are:

$$\left.\begin{array}{l} \sum_1^{r-2}(x_iy_k-x_ky_i)^2=0, \quad \sum_1^{r-2}(x_{r+1}y_i-x_iy_{r+1})y_i=0, \\ x_{r+s-2}y_{r+s'-2}-x_{r+s'-2}y_{r+s-2}=0, \quad s'\neq s, \quad u=0. \end{array}\right\} \quad (97)$$

The surface (96a) *is the locus of* ∞^{r-2} *flats which intersect a family of* ∞^{n-r} *flats.* Such a locus we shall call a *flat-regulus* and denote it by the symbol $R^{(r-2)}_{\frac{n-6}{2}+r}$, the lower subscript indicating the dimensions of the surface. If a regulus is a quartic $n-2$-spread, or is the intersection of two or more such spreads, we shall call it a *quartic regulus;* it may of course happen, as in the case of (96a) and (96b), that one or more, but not all, reduce to cubic spreads, quadrics or even flats. A cubic (quadric) regulus is a cubic (quadric) $n-2$-spread, or the intersection of two or more such spreads. (96a), (96b) and (90′) are quartic reguli, and (90) a quadric regulus. In 5-space, however, $n=6$, the reguli (96a) and (96b) are cubic, as is also the case in any odd space if $r=3$; the regulus has then only a single infinity of flats.

The flats generating the surfaces belong to certain systems of linear flat-complexes which we shall study in a more comprehensive manner in another paper. We shall state the results obtained in the following theorem which is a generalization of the corresponding one for 3-space given by Lie. It is pertinent to point out here the great generality of the theorem, since in 3-space only one type can exist, namely, $r=1$ for cyclides of the third order, and $r=3$ for those of the fourth:

THEOREM. *The generalized flat-sphere transformation* (3) *carries the Dupin cyclide of the third order and type* $r \leqq \frac{n-2}{2}$ *in* S_{n-1} ($n-1$ odd) *into a quadric and quartic regulus of* $\frac{n-2}{2}+r$ *dimensions which are conjugate to each other. If* $r=1$ *the quartic regulus is cubic, and in* 5*-space it reduces to a quadric regulus. If* $r=\frac{n-2}{2}$ *the transform is a quartic* $n-2$*-spread. The*

same transformation carries the Dupin cyclide of the fourth order and type $r \leqq \frac{n+2}{2}$ *into two conjugate quartic reguli of dimensions* $\frac{n+2}{2}+r$, *which become cubic if* $r=3$. *In* 5*-space they are quadric reguli. If* $r=\frac{n+2}{2}$ *the transform is a quartic* $n-2$*-spread which in* 5*-space has a quadric locus of double-points at infinity.*

The asymptotic directions proceeding from a point on a regulus $R^{(r-2)}_{\frac{n-6}{2}+r}$ lie in the two flats that intersect at the point. There are $\infty^{\frac{n-4}{2}}$ directions which lie in the flat of the first set, and ∞^{r-3} lying in that of the second set. The asymptotic lines are therefore indeterminate to a less extent than the lines of curvature on a cyclide.* For $n=4$ $r=3$ (ordinary space) the number of directions are finite and equal to 2. The group of $\infty^{\frac{n+1\cdot n+2}{2}}$ contact-transformations whose characteristic functions are given by (2), carries flats into flats, and it therefore transforms all flat-reguli *inter se*. The group is therefore related to all the spreads $R^{(r)}_{\frac{n-2}{2}+r}$ and $R^{(r-2)}_{\frac{n-6}{2}+r}$ in the same way that the projective group of ∞^{15} transformations are related to all the quadric surfaces in 3-space. We have thus obtained a class of surfaces in odd space which from the standpoint of Sphere-Flat Geometry is the generalization of the quadric surface in 3-space, and which we shall meet with again in the study of flat-complexes.

§ 4. *The Asymptotic Lines on a Sphere in Odd Space.*

We have seen that the asymptotic lines on the transform of cyclides in S_{n-1} are to a certain extent indefinite; these spreads are therefore exceptions to the general rule. Thus, for example, the transforms of homofocal or co-axial quadrics have definite asymptotic lines.

The asymptotic lines on a quadric surface have not been determined except for 3-space. Let the quadric be written

$$\Sigma a_i x^2 + \Sigma b_i y^2 + c z^2 = 1. \tag{98}$$

Transforming by Euler's transformation (79) we have the surface

$$F^2 = \frac{1}{c}\left[1 + c\Sigma \frac{\alpha_i^2}{a_i}\right][1 - \Sigma b_i \beta_i^2], \tag{98'}$$

* This is not true relatively. The regulus is a surface of dimensions $\frac{n-6}{2}+r$ while the corresponding cyclide is of $n-2$ dimensions.

so that the tangential equation of the transform in the α_i, β_i coordinates (5) is known, and the parametric equations of the surface may be derived from it. The surface is not real, and the determination of the lines of curvature is not a simple matter. In the case of a sphere a direct method is successful. Let the sphere be written

$$\Sigma x_i^2+\Sigma y_i^2+z^2=r^2; \tag{99}$$

the system of total differential equations which determine the lines, viz.:

$$dx_i dp_{\frac{n-2}{2}}+dy_i dq_{\frac{n-2}{2}}=0, \quad dq_i dp_{\frac{n-2}{2}}-dp_i dq_{\frac{n-2}{2}}=0, \qquad i=1, 2, \ldots, \frac{n-2}{2}, \tag{100}$$

may be replaced by the following:

$$dx_i dp_i+dy_i dq_i=0, \quad dq_1 dp_i-dp_1 dq_i=0, \qquad i=1, 2, \ldots, \frac{n-2}{2}. \tag{100'}$$

We have from (99),

$$p_i=-\frac{x_i}{z}, \quad q_i=-\frac{y_i}{z}.$$

Introducing the values of dp_i and dq_i in (100′), and keeping account of the relations $dx_1 dy_i-dy_1 dx_i=0$, $\left(i=2, 3, \ldots, \frac{n-2}{2}\right)$, the system becomes

$$\left.\begin{aligned} dz(x_i dx_i+y_i dy_i) &= z(dx_i^2+dy_i^2), \\ dz^2(y_1 x_i-x_1 y_i) &= dz(y_1 dx_i-x_1 dy_i+x_i dy_1-y_i dx_1). \end{aligned}\right\} \tag{101}$$

We shall first suppose that $dz \neq 0$. The second set may then be integrated at once giving $\frac{n-4}{2}$ integrals:

$$y_1 x_i-x_1 y_i=2c_{i-1}z, \qquad i=2, 3, \ldots, \frac{n-2}{2}. \tag{102}$$

In order to integrate the first set we shall introduce the variables u_i and v_i, putting

$$\left.\begin{aligned} x_i+iy_i&=u_i, \quad x_i=\frac{u_i+v_i}{2}, \\ x_i-iy_i&=v_i, \quad y_i=\frac{u_i-v_i}{2_i}, \end{aligned}\right\} \tag{103}$$

so that (90) and (102) become

$$z^2+\Sigma u_i v_i=r^2, \quad u_1 v_i-v_1 u_i=2c_{i-1}z, \qquad i=2, 3, \ldots, \frac{n-2}{2}, \tag{104}$$

and the first set (101) is now

$$\frac{dz}{z}=\frac{2du_i dv_i}{v_i du_i+u_i dv_i},$$

which, since $dv_1du_i - du_1dv_i = 0$, may be written in the form:

$$\frac{du_i}{dz} = \frac{v_i du_1 + u_i dv_1}{2z dv_1}.$$

Introducing now a factor of proportionality ρ, we may write this system in the final form:

$$\rho du_i = u_i dv_1 + v_i du_1, \quad \rho dz = 2z dv_1. \tag{105}$$

Differentiating (104) we have,

$$u_1 dv_i + v_i du_1 - v_1 du_i - u_i dv_1 = 2c_{i-1} dz, \quad \Sigma(u_i dv_i + v_i du_i) = -2z dz, \tag{106}$$

which, together with (105), constitute a system of n linear and homogeneous equations in $n-1$ unknowns du_i, dv_i, dz. This system will be consistent if, and only if, the determinant of the system vanishes, *i. e.*, if

$$\begin{vmatrix} v_1-\rho & 0 & 0 \ldots & 0 & u_1 & 0 & 0 \ldots 0 \\ v_2 & -\rho & 0 \ldots & 0 & u_2 & 0 & 0 \ldots 0 \\ v_3 & 0 & -\rho \ldots & 0 & u_3 & 0 & 0 \ldots 0 \\ \cdot & \cdot & \cdot\ldots & \cdot & \cdot & \cdot & \cdot\ldots \\ \cdot & \cdot & \cdot\ldots & \cdot & \cdot & \cdot & \cdot\ldots \\ 0 & 0 & 0 \ldots & -\rho & 2z & 0 & 0 \ldots 0 \\ v_2 & -v_1 & 0 \ldots & -2c_1 & -u_2 & u_1 & 0 \ldots 0 \\ v_3 & 0 & -v_1 \ldots & -2c_2 & -u_3 & 0 & u_1 \ldots 0 \\ \cdot & \cdot & \cdot\ldots & \cdot & \cdot & \cdot & \cdot\ldots \\ \cdot & \cdot & \cdot\ldots & \cdot & \cdot & \cdot & \cdot\ldots \\ v_1 & v_2 & v_3 \ldots & 2z & u_1 & u_2 & u_3 \ldots u_{\frac{n-2}{2}} \end{vmatrix} = 0. \tag{107}$$

Expanding this determinant we find that it is a quadratic in ρ, $\rho^{\frac{n}{2}-2}$ being a factor. The zero value of the root may be neglected as no solution of the system (105) corresponds to it. The quadratic equation is:

$$\rho^2 + \frac{4z(u_1 z + \sum_2^{\frac{n-2}{2}} c_{i-1}u_i)}{\Sigma u_i^2}\rho + \frac{4z^2[\sum_1^{\frac{n-2}{2}} c_{i-1}^2 - u_1 v_1]}{\Sigma u_i^2} = 0. \tag{108}$$

We now solve the equations (104) for v_1 after eliminating $v_2, v_3, \ldots, v_{\frac{n-2}{2}}$ and substitute in (108). Solving we have the two roots

$$\rho = \frac{-2z(u_1 z + \sum_2^{\frac{n-2}{2}} c_{i-1}u_i) \pm 2z\sqrt{R}}{\Sigma u_i^2},$$

where $R = \sqrt{u_1^2 r^2 - \Sigma c_{i-1}^2 \sum_1 u_i^2 + (\sum_2 c_{i-1}u_i)^2}$. Introducing ρ in the system (105),

and eliminating the v's by means of (108), we have the following system of differential equations:

$$\frac{dz}{u_1(r^2+z^2)\mp 2z\sqrt{R}}=\frac{du_1}{u_1(u_1z+\sum\limits_2^{\frac{n-2}{2}}c_{i-1}u_i\mp\sqrt{R})}$$

$$=\frac{du_2}{u_2(u_1z+\sum\limits_2^{\frac{n-2}{2}}c_{i-1}u_i\mp\sqrt{R})-c_1\Sigma u_i^2}=\ldots.$$

$$=\frac{du_{\frac{n-2}{2}}}{u_{\frac{n-2}{2}}(u_1z+\sum\limits_2^{\frac{n-2}{2}}c_{i-1}u_i\mp\sqrt{R})-c_{\frac{n-4}{2}}\Sigma u_i^2}. \tag{109}$$

By properly combining we easily find the following algebraic integrals:

$$\frac{u_{i+2}}{c_{i+1}}=\frac{u_2}{c_1}+k_iu_1,\qquad i=1,2,\ldots,\frac{n-6}{2}, \tag{110}$$

the k's being constants of integration. The remaining integrals may now be found as follows: We substitute the values of the u's from (110) in the expression for R, above, and find

$$R=\sqrt{r^2-(1+\sum\limits_1^{\frac{n-6}{2}}k_i^2)\sum\limits_1^{\frac{n-4}{2}}c_i^2+(\sum\limits_1^{\frac{n-6}{2}}k_ic_{i+1})^2u_1}=Au_1;$$

we also have

$$\sum\limits_2^{\frac{n-2}{2}}c_{i-1}u_i=\sum\limits_1^{\frac{n-6}{2}}k_ic_{i+1}u_1+\frac{\Sigma c_i^2}{c_1}u_2=Bu_1+Cu_2,$$

$$\sum\limits_1^{\frac{n-4}{2}}u_i^2=(1+\Sigma k_i^2)u_1^2+\frac{2\Sigma k_ic_{i+1}}{c_1}u_1u_2+\frac{\Sigma c_i^2}{c_1^2}u_2^2$$

$$=Du_1^2+\frac{2B}{c_1}u_1u_2+\frac{C}{c_1}u_2^2,\quad r^2-A^2=c_1CD-B^2.$$

We have then to integrate the equations

$$\frac{dz}{u_1(r^2+z^2)\mp 2zAu_1}=\frac{du_1}{u_1(u_1z+Bu_1+Cu_2\mp Au_1)}$$

$$=\frac{du_2}{u_2(u_1z+Bu_1+Cu_2\mp Au_1)-c_1\left(Du_1^2+\frac{2B}{c_1}u_1u_2+\frac{C}{c_1}u_2^2\right)},$$

which may be further simplified by putting $u_2 = vu_1$:

$$\frac{dz}{r^2+z^2\mp 2Az} = \frac{du_1}{u_1(z+B+Cv\mp A)} = -\frac{dv}{c_1D+2Bv+Cv^2}. \tag{111}$$

These equations may be integrated by quadratures. We find,

$$z = \frac{\alpha(r^2-A^2)-(Cv+B)}{1+\alpha(Cv+B)} \pm A, \quad u_1 = \frac{\beta}{1+\alpha(Cv+B)} u_2 = \frac{\beta v}{1+\alpha(Cv+B)}. \tag{112}$$

The remaining u's and all the v's are now obtained from the equations (110) and (104). A rather long but not difficult calculation shows that they may be expressed in the form:

$$u_{i+2} = \frac{p_{i+2}+q_{i+2}v}{1+\alpha(Cv+B)}, \quad v_i = \frac{r_i+s_iv}{1+\alpha(Cv+B)}, \qquad i=1, 2, \ldots, \frac{n-2}{2}, \tag{113}$$

where the coefficients p_i, q_i, r_i, s_i depend on the $n-3$ integration constants c_i, k_i, α, β. *From these equations it appears at once that the integral curves are straight lines.* Through any point on the sphere will pass two such lines corresponding to the two roots of (108), or, corresponding to the + and − sign of A in (111).

We shall investigate the remaining asymptotic loci. We assumed $dz \neq 0$. Let $dz=0$. The second set of equations (101) vanish identically, and the equations of the first set are:

$$dz=0, \quad dx_i^2+dy_i^2=0, \qquad i=1, 2, \ldots, \frac{n-2}{2},$$

which are satisfied if we put

$$\text{I.} \quad z=C, \quad x_i+iy_i=m_i, \quad \text{or,} \quad \text{II.} \quad z=C, \quad x_i-iy_i=n_i.$$

The first set of flats intersect the sphere in an $\frac{n-4}{2}$ dimensional flat whose equations are:

$$z=C, \quad x_i+iy_i=m_i, \quad C^2+\Sigma m_i(x_i-iy_i)=r^2,$$

and the second set intersects the sphere in a similar flat whose equations are:

$$z=C, \quad x_i-iy_i=n_i, \quad C^2+\Sigma n_i(x_i+iy_i)=r^2.$$

Through any point on the sphere pass two such flats and hence there will be $\infty^{\frac{n-4}{2}}$ asymptotic directions through it and lying in the respective flats. We shall call these loci *asymptotic flats.* The results obtained will now be stated in the following

THEOREM. *There are two sets of asymptotic loci on a sphere in odd-dimensional space. The first set consists of a double family of* ∞^{n-3} *straight*

lines of which through every point on the sphere will pass two. The second set consists of a double family of $\infty^{\frac{n}{2}}$ $\frac{n-4}{2}$*-dimensional flats, the asymptotic flats, and through every point on the sphere will pass two flats and* $\infty^{\frac{n-4}{2}}$ *asymptotic directions lying in each flat.*

For 5-space these flats are straight lines so that we may say:

Through every point on a sphere in 5-space pass four definite asymptotic directions and the four sets of asymptotic lines consist of a four-fold family of ∞^3 *straight lines.*

In 3-space the first set of ∞^1 asymptotic lines remain while the asymptotic flats, being of zero dimensions, become the ∞^2 points of the sphere.

There exist an indefinite number of surfaces whose asymptotic loci are straight lines and flats, namely, all the transforms of the sphere by the transformations of the group (2). Thus, the quadrics

$$\Sigma a_i(x_i^2+y_i^2)+cz^2=r^2$$

belong to this class, since the transformation,

$$x_i=a_ix_i', \quad y_i=a_iy_i', \quad z=cz',$$

carries flats into flats.

The transform of the sphere in $\bar{\bar{S}}_{n-1}$ is a sextic surface in S_{n-1} whose equation is:

$$[r^2(1+X_{n-1}^2)-\Sigma(X_{2i-1}+iX_{2i})^2][1+X_{n-1}^2+\Sigma(X_{2i-1}-iX_{2i})^2]=X_{n-1}^2(1+\sum_1^{n-1}X_i^2)^2,$$

which has the absolute as locus of double points. The surface has four focal sheets if $n \geqq 6$. *In 5-space the surface has circular lines of curvature in all four systems;* it therefore presents a striking resemblance to the cyclides in ordinary space.

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